

# NEF VECTOR BUNDLES ON A PROJECTIVE SPACE WITH FIRST CHERN CLASS 3 AND SECOND CHERN CLASS LESS THAN 8

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**ABSTRACT.** We classify nef vector bundles on a projective space with first Chern class three and second Chern class less than eight over an algebraically closed field of characteristic zero; we see, in particular, that, except for one possibly occurring case in dimension three, these nef vector bundles are globally generated. We also give an example of nef but non-globally generated vector bundles with first Chern class three and second Chern class eight on a projective plane.

## 1. INTRODUCTION

Let  $\mathcal{E}$  be a nef vector bundle of rank  $r$  on a projective space  $\mathbb{P}^n$  over an algebraically closed field  $K$  of characteristic zero. Let  $c_1$  be the first Chern class of  $\mathcal{E}$ . Then  $c_1$  is non-negative since  $\mathcal{E}$  is nef. In [PSW92, Theorem 1], Peternell-Szurek-Wiśniewski classified such  $\mathcal{E}$ 's in case  $c_1 \leq 2$ , based on the study [SW90] of Szurek-Wiśniewski. If  $c_1 \leq 2$  and  $n \geq 2$ , then  $\mathbb{P}(\mathcal{E})$  is a Fano manifold, and their proof is based on analysis of contraction morphisms of extremal rays. In [Ohn14, §6], a different proof of the classification in case  $c_1 \leq 2$  is given, based on analysis of some twist  $\mathcal{E}(d)$  of  $\mathcal{E}$  with the full strong exceptional sequence  $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$  of line bundles.

In this paper, we continue our approach to the classification of nef vector bundles in the next case  $c_1 = 3$ , and we obtain the following result. Note here that if  $c_1 = 3$  then the anti-canonical bundle of  $\mathbb{P}(\mathcal{E})$  is nef if  $n \geq 2$  and ample if  $n \geq 3$ . Moreover if  $c_1 = 3$  then  $0 \leq c_2 \leq c_1^2 = 9$  where  $c_2$  denote the second Chern class of  $\mathcal{E}$ .

**Theorem 1.1.** *Let  $\mathcal{E}$  be a nef vector bundle of rank  $r$  on a projective space  $\mathbb{P}^n$ . Suppose that  $c_1 = 3$ . Then  $c_2 \leq 7$  if and only if  $c_2$  and  $\mathcal{E}$  satisfy one of the following:*

- (1)  $c_2 = 0$  and  $\mathcal{E} \cong \mathcal{O}(3) \oplus \mathcal{O}^{\oplus r-1}$ ;
- (2)  $c_2 = 2$  and  $\mathcal{E} \cong \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2}$ ;
- (3)  $c_2 = 3$  and  $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r-3}$ ;
- (4)  $c_2 = 3$ ,  $n = 2$ , and  $\mathcal{E} \cong T_{\mathbb{P}^2} \oplus \mathcal{O}^{\oplus r-2}$ ;

(In the following,  $\mathcal{E}$  fits in one of the following exact sequences.)

- (5)  $c_2 = 3$  and  $0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r} \rightarrow \mathcal{E} \rightarrow 0$ ;
- (6)  $c_2 = 4$  and  $0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-1} \rightarrow \mathcal{E} \rightarrow 0$ ;
- (7)  $c_2 = 4$ ,  $n = 3$ , and  $0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 4} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+2} \rightarrow \mathcal{E} \rightarrow 0$ ;
- (8)  $c_2 = 4$ ,  $n = 4$ , and

$$0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O}(-2)^{\oplus 5} \rightarrow \mathcal{O}(-1)^{\oplus 10} \rightarrow \mathcal{O}^{\oplus r+6} \rightarrow \mathcal{E} \rightarrow 0;$$

- (9)  $c_2 = 5$  and  $0 \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E} \rightarrow 0$ ;

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$$(10) \ c_2 = 5, \ n = 3 \text{ or } 4, \text{ and } 0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 5} \rightarrow \mathcal{O}^{\oplus r+4} \rightarrow \mathcal{E} \rightarrow 0;$$

$$(11) \ c_2 = 5, \ n = 3, \ h^0(\mathcal{E}(-1)) = 1, \ h^1(\mathcal{E}(-1)) = 2, \text{ and}$$

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-2)^{\oplus 2} \rightarrow \mathcal{O}(-1)^{\oplus 4r+6} \oplus T_{\mathbb{P}^3}(-3)^{\oplus 2} \\ \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(-1)^{\oplus 4r+10} \rightarrow \mathcal{E} \rightarrow 0; \end{aligned}$$

$$(12) \ c_2 = 6 \text{ and } 0 \rightarrow \mathcal{O}(-1)^{\oplus 3} \rightarrow \mathcal{O}^{\oplus r+3} \rightarrow \mathcal{E} \rightarrow 0;$$

$$(13) \ c_2 = 6 \text{ and } 0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r} \rightarrow \mathcal{E} \rightarrow 0;$$

$$(14) \ c_2 = 7 \text{ and } 0 \rightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus r+2} \rightarrow \mathcal{E} \rightarrow 0.$$

It is uncertain whether the case (11) in Theorem 1.1 really occurs or not. If it happens, we would infer that there exists a nef but non-globally generated vector bundles in case  $(n, c_1, c_2) = (3, 3, 5)$ , thanks to the classification of globally generated vector bundles with  $c_1 = 3$  by Anghel-Manolache [AM13] and Sierra-Ugaglia [SU14]. On the other hand, other cases really occur: in the case (7), examples of  $\mathcal{E}$  are  $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-4} \oplus \Omega_{\mathbb{P}^3}(2)$  and  $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-3} \oplus \mathcal{N}(1)$  where  $\mathcal{N}$  is a null correlation bundle on  $\mathbb{P}^3$  (see Remark 7.1); in the case (8),  $\mathcal{E}$  is given by a locally free resolution in terms of  $\mathcal{O}(-3)$ -twist of the full strong exceptional sequence  $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(4)$  in accordance with [Ohn14], but  $\mathcal{E}$  is in fact isomorphic to  $\Omega_{\mathbb{P}^4}(2) \oplus \mathcal{O}^{\oplus r-4}$  (see Remark 7.2); in the case (10), if  $n = 4$ ,  $\mathcal{E}$  is nothing but an extension of the Tango bundle [Tan76] (see also [OSS80, Chap. I §4.3]) on  $\mathbb{P}^4$  by a trivial bundle  $\mathcal{O}^{\oplus r-3}$ , so that  $\Omega_{\mathbb{P}^4}^2(3)$  is a typical example (see Remark 8.1), and if  $n = 3$ , the restriction of such to a hyperplane  $\mathbb{P}^3$  is an example.

As in the proof in the case where  $c_1 \leq 2$  in [Ohn14], the main feature of our proof of Theorem 1.1 is an application of the spectral sequence deduced in [OT14, Theorem 1] from Bondal's theorem [Bon89, Theorem 6.2]. Besides this spectral sequence, some of key ingredients of our proof are the Riemann-Roch formula, the Kawamata-Viehweg vanishing theorem [Kaw82] [Vie82], and the non-negativity of Chern classes of nef vector bundles.

Note that, except for the case (11) in Theorem 1.1, nef vector bundles with  $c_1 = 3$  and  $c_2 < 8$  are all globally generated. On the contrary to this case, in the case where  $c_2 = 8$ , we can construct explicitly an example of a nef but non-globally generated vector bundle on a projective plane; more precisely we obtain the following

**Proposition 1.2.** *Given an integer  $r \geq 2$  and a point  $w$  in a projective plane  $\mathbb{P}^2$ , there exists a vector bundle  $\mathcal{E}$  fitting in an exact sequence*

$$0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E} \rightarrow k(w) \rightarrow 0$$

where  $k(w)$  denotes the residue field of the point  $w$ . Moreover a vector bundle  $\mathcal{E}$  fitting in the sequence above is nef but non-globally generated with  $c_1 = 3$  and  $c_2 = 8$ .

As one of related results of Theorem 1.1 and Proposition 1.2, besides Anghel-Manolache [AM13] and Sierra-Ugaglia [SU14] mentioned above, we note that Langer [Lan98] classified smooth Fano 4-folds with adjunction theoretic scroll structure and  $b_2 = 2$ ; his classification includes that of nef and big rank 2 bundles with  $c_1 = 3$  on  $\mathbb{P}^2$  and  $\mathbb{P}^3$ .

The content of this paper is as follows. In §2, we recall Bondal's theorem [Bon89, Theorem 6.2] and its related results including the spectral sequence deduced in [OT14, Theorem 1]. The results in §2 are fundamental throughout this paper. In §3, we begin with our proof of Theorem 1.1; based on results in [Ohn14], we reduce the problem to the case where  $H^0(\mathcal{E}(-2)) = 0$ . We also show in §3 that this reduction enable us to

assume that  $3 \leq c_2 \leq 7$ . Several other formulas—such as the Riemann-Roch formulas—used repeatedly in this paper are also presented in §3. In §4, we collect some lemmas used several times in this paper. In §5, we give a key lemma, Lemma 5, for the case  $n = 2$ . Since our proof of Theorem 1.1 goes by induction on  $n$ , Lemma 5 in §5 together with exact sequences (3.15) and (3.16) in §3.1 is crucial to the whole proof; in fact, the restriction to the case  $c_2 \leq 7$  derives from Lemma 5. In §§6, 7, 8, 9, and 10, we give a proof of Theorem 1.1 in case  $c_2 = 3, 4, 5, 6$ , and  $7$  respectively. In §11, we show that, once we establish the global generation for the case  $n = 2$ , we can prove the global generation for  $n \geq 3$  unless  $(n, c_2, c_3) = (3, 5, 3)$  without a detailed description of vector bundles; based on this result, we can invoke the classification given in [AM13] or [SU14] to prove Theorem 1.1, but we take a unified and consistent approach to provide a full proof; our proof is different from those of [AM13] or [SU14]. In §12, we give a proof of Proposition 1.2. Finally note that, since globally generated vector bundles are nef, some properties of globally generated vector bundles also hold more generally for nef vector bundles, but some do not; in §13, we give two examples which illustrate that a typical exact sequence for globally generated vector bundles—related to degeneracy locus of general global sections—does not necessarily exist for nef vector bundles.

In a forthcoming paper [Ohn17], we shall classify nef vector bundles on a projective space with  $c_1 = 3$  and  $c_2 = 8$ : we shall show that such bundles exist only on a projective plane and every such bundle fits in the exact sequence given in Proposition 1.2.

**1.1. Notation and conventions.** Throughout this paper we work over an algebraically closed field  $K$  of characteristic zero. Basically we follow the standard notation and terminology in algebraic geometry. For a vector bundle  $\mathcal{E}$ ,  $\mathbb{P}(\mathcal{E})$  denotes  $\text{Proj } S(\mathcal{E})$ , where  $S(\mathcal{E})$  denotes the symmetric algebra of  $\mathcal{E}$ . The tautological line bundle  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  is also denoted by  $H(\mathcal{E})$ . For a coherent sheaf  $\mathcal{F}$  on a smooth projective variety  $X$ , we denote by  $c_i(\mathcal{F})$  the  $i$ -th Chern class of  $\mathcal{F}$ . We say that a vector bundle is (non-)globally generated if it is (not) generated by global sections. For coherent sheaves  $\mathcal{F}$  and  $\mathcal{G}$  on  $X$ ,  $h^q(\mathcal{F})$  denotes  $\dim H^q(\mathcal{F})$ , and  $\text{hom}(\mathcal{F}, \mathcal{G})$  denotes  $\dim \text{Hom}(\mathcal{F}, \mathcal{G})$ . Finally we refer to [Laz04] for the definition and basic properties of nef vector bundles.

## 2. PRELIMINARIES

In our proof of Theorem 1.1, we shall apply repeatedly a spectral sequence deduced in [OT14, Theorem 1]. So we shall recall this sequence in this section. Note that this sequence is a corollary of Bondal's theorem [Bon89, Theorem 6.2] as can be seen below.

Let  $X$  be a smooth projective variety over  $K$ ,  $D^b(X)$  the bounded derived category of the abelian category of coherent sheaves on  $X$ . Assume that there exists a full strong exceptional sequence  $G_0, \dots, G_m$  in  $D^b(X)$ , and let  $G$  be the direct sum  $\bigoplus_{j=0}^m G_j$  of  $G_0, \dots, G_{m-1}$ , and  $G_m$  in  $D^b(X)$ .

Recall that if  $X = \mathbb{P}^n$  then  $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$  is a full strong exceptional sequence in  $D^b(\mathbb{P}^n)$  by Beilinson's theorem [Bei78, Theorem].

Let  $A$  be the endomorphism ring  $\text{End}_{D^b(X)}(G)$  of  $G$ , and let  $e_j$  be the composite of the projection  $G \rightarrow G_j$  and the inclusion  $G_j \rightarrow G$ . Then  $e_j \in A$ . Define a right  $A$ -module  $P_j$  by  $P_j = e_j A$ . The natural isomorphism  $A \cong \bigoplus_{j=0}^m P_j$  of right  $A$ -modules implies that  $P_j$  is a projective right  $A$ -module. Let  $S_j$  be the simple right  $A$ -module such that  $S_j e_j \cong K$  and  $S_j e_k \cong 0$  for all  $k \neq j$  and  $0 \leq k \leq m$ .

Let  $D^b(\text{mod } A)$  be the bounded derived category of the abelian category  $\text{mod } A$  of finitely generated right  $A$ -modules. Bondal's theorem [Bon89, Theorem 6.2] states that  $\text{RHom}(G, \bullet) : D^b(X) \rightarrow D^b(\text{mod } A)$  is an exact equivalence. Since  $\bullet \otimes^L G : D^b(\text{mod } A) \rightarrow D^b(X)$  is a quasi-inverse of  $\text{RHom}(G, \bullet)$ , we have a natural isomorphism

$$\text{RHom}(G, \bullet) \otimes^L G \cong \bullet$$

of functors on  $D^b(X)$ . If we write down this isomorphism for a coherent sheaf  $F$  on  $X$  in terms of a spectral sequence, we obtain the following spectral sequence ([OT14, Theorem 1])

$$(2.1) \quad E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, F), G) \Rightarrow E^{p+q} = \begin{cases} F & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

We call this sequence the Bondal spectral sequence. In practice, in order to apply the Bondal spectral sequence (2.1), we need to compute  $E_2^{p,q}$ . One way to compute  $E_2^{p,q}$  is by definition:  $E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, F), G)$ , i.e., through a projective resolution of the right  $A$ -module  $\text{Ext}^q(G, F)$ ; recall here (see [Ohn14, Lemma 2.1] for a proof) that a finitely generated right  $A$ -module  $\text{Ext}^q(G, F)$  has a projective resolution of the form

$$(2.2) \quad 0 \rightarrow P_0^{\oplus e_{m,0}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^{m-l} P_j^{\oplus e_{l,j}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^m P_j^{\oplus e_{0,j}} \rightarrow \text{Ext}^q(G, F) \rightarrow 0$$

where  $e_{0,j} = \dim \text{Ext}^q(G_j, F)$  for  $0 \leq j \leq m$  and  $e_{l,j}$  is determined inductively for  $l \geq 1$  and  $j \leq m-l$  by the following formula:  $e_{l,j} = \sum_{j < k} e_{l-1,k} \text{hom}(G_j, G_k)$ . We shall freely use the following isomorphism:

$$P_j \otimes_A^L G = P_j \otimes_A G \cong G_j.$$

This isomorphism together with (2.2) implies that  $E_2^{p,q}$  is the  $(-p)$ -th homology of the following complex

$$0 \rightarrow G_0^{\oplus e_{m,0}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^{m-l} G_j^{\oplus e_{l,j}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^m G_j^{\oplus e_{0,j}} \rightarrow 0.$$

In this paper, we shall apply the Bondal spectral sequence (2.1) with  $X = \mathbb{P}^n$ ,  $m = n$ , and  $G_j = \mathcal{O}(j)$  for  $0 \leq j \leq n$ .

Throughout this paper, we fix the notation:  $G_j = \mathcal{O}(j)$  for  $0 \leq j \leq n$ ;  $G = \bigoplus_{j=0}^m G_j$ ;  $A = \text{End}_{D^b(\mathbb{P}^n)}(G)$ ;  $P_j$  and  $S_j$  for  $0 \leq j \leq n$  as above.

A typical projective resolution of the form (2.2) in this paper is in the case where  $q = 0$  and  $F = \mathcal{E}(d)$  for some non-negative integer  $d$ . For example, if  $\text{hom}(\mathcal{O}(2), \mathcal{E}(d)) = 0$ , we frequently and sometimes implicitly consider a projective resolution of the form

$$(2.3) \quad 0 \rightarrow P_0^{\oplus (n+1)e_{0,1}} \rightarrow P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}(d)) \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(d))$  and  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(d))$ .

Finally note that the Bott formula [OSS80, p. 8] implies  $\text{RHom}(G, \Omega_{\mathbb{P}^n}^j(j)) \cong S_j[-j]$  for  $0 \leq j \leq n$ . Hence we have isomorphisms

$$(2.4) \quad S_j \otimes_A^L G \cong \Omega_{\mathbb{P}^n}^j(j)[j].$$

for  $0 \leq j \leq n$ . In particular,

$$(2.5) \quad S_n \otimes_A^L G \cong \mathcal{O}(-1)[n].$$

Note that this gives another way to compute  $E_2^{p,q} = \mathcal{H}^p(\text{Ext}^q(G, \mathcal{E}(d)) \otimes_{\mathcal{L}_A} G)$ : through a filtration of  $\text{Ext}^q(G, \mathcal{E}(d))$  with subquotients the direct sums of the simple modules  $S_j$ . For example, if  $\text{Ext}^q(G, \mathcal{E}(d)) \cong S_j$  then  $E_2^{p,q} = \mathcal{H}^p(\Omega_{\mathbb{P}^n}^j(j)[j])$ , and thus  $E_2^{p,q} = 0$  if  $p \neq -j$  and  $E_2^{-j,q} = \Omega_{\mathbb{P}^n}^j(j)$ . In particular, if  $\text{Ext}^q(G, \mathcal{E}(d)) \cong S_n$  then  $E_2^{-n,q} = \mathcal{O}(-1)$  and  $E_2^{p,q} = 0$  if  $p \neq -n$ . We shall also use these formulas frequently.

### 3. SET-UP AND FORMULAS FOR THE PROOF OF THEOREM 1.1

After some preparatory considerations, our proof of Theorem 1.1 shall be divided by the values of the second Chern class  $c_2$  and dimension  $n$ . In addition, several formulas shall also be applied repeatedly throughout this paper; in this section, we give some preparatory part of our proof, and we also collect several formulas needed later.

Let  $\mathcal{E}$  be a nef vector bundle of rank  $r$  on a projective space  $\mathbb{P}^n$  with  $c_1 = 3$  over an algebraically closed field  $K$  of characteristic zero. We begin with our proof by dealing with the case  $\text{Hom}(\mathcal{O}(3), \mathcal{E}) \neq 0$ . Suppose that  $\text{Hom}(\mathcal{O}(3), \mathcal{E}) \neq 0$ . Then  $\mathcal{E} \cong \mathcal{O}(3) \oplus \mathcal{O}^{\oplus r-1}$  by [Ohn14, Proposition 5.2 and Remark 5.3]. This is the case (1) of Theorem 1.1.

In the rest of our proof, we always assume that  $\text{Hom}(\mathcal{O}(3), \mathcal{E}) = 0$ . Hence  $r \geq 2$ .

Suppose that  $\text{Hom}(\mathcal{O}(2), \mathcal{E}) \neq 0$ . Then it follows from [Ohn14, Theorem 6.4] that  $\mathcal{E}$  is in the case (2) or (5) of Theorem 1.1.

In the rest of our proof, we always assume that

$$(3.1) \quad H^0(\mathcal{E}(-2)) = \text{Hom}(\mathcal{O}(2), \mathcal{E}) = 0.$$

Recall that the Kodaira vanishing theorem implies that

$$(3.2) \quad H^q(\mathcal{E}|_{L^l}(3-k)) = 0$$

for all  $q > 0$ , all  $l$ -dimensional linear subspace  $L^l \subseteq \mathbb{P}^n$ , and all  $k \leq l$ , since  $\mathcal{E}$  is a nef vector bundle with  $c_1 = 3$  (see [Ohn14, Lemma 4.1 (1)] for a proof). Moreover, if  $H(\mathcal{E}|_{L^l})$  is big in addition, then the Kawamata-Viehweg vanishing theorem implies that

$$(3.3) \quad H^q(\mathcal{E}|_{L^l}(2-k)) = 0$$

for all  $q > 0$  and all  $k \leq l$  (see [Ohn14, Lemma 4.1 (2)] for a proof).

Since  $\mathcal{E}$  is nef,  $H^q(\mathcal{E}|_{L^1}(1)) = 0$  for all  $q > 0$  and all line  $L^1$  in  $\mathbb{P}^n$ . Together with (3.2), this implies that  $H^2(\mathcal{E}|_{L^2}) = 0$  for any plane  $L^2 \subseteq \mathbb{P}^n$ . The vanishing  $H^2(\mathcal{E}|_{L^2}) = 0$  then implies that  $H^2(\mathcal{E}|_{L^2}(-1)) = 0$  since  $H^q(\mathcal{E}|_{L^1}) = 0$  for all  $q > 0$ . Moreover we have  $H^2(\mathcal{E}|_{L^2}(-2)) = 0$  since  $H^q(\mathcal{E}|_{L^1}(-1)) = 0$  for all  $q > 0$ . Summing up, we have

$$(3.4) \quad H^2(\mathcal{E}|_{L^2}(-k)) = 0$$

for all  $k \leq 2$  and any plane  $L^2$  in  $\mathbb{P}^n$ .

The Riemann-Roch formula for a twisted vector bundle  $\mathcal{E}(m)$  on  $\mathbb{P}^2$  is

$$\chi(\mathcal{E}(m)) = \frac{1}{2}\{c_1^2 + (2m+3)c_1 + rm(m+3)\} + r - c_2(\mathcal{E})$$

where  $c_1 = c_1(\mathcal{E})$ . Since  $c_1 = 3$ , the formula above is reduced to the following:

$$(3.5) \quad \chi(\mathcal{E}(m)) = \frac{1}{2}(rm+6)(m+3) + r - c_2(\mathcal{E}).$$

In particular, we have  $\chi(\mathcal{E}(-2)) = 3 - c_2(\mathcal{E})$ . The vanishing (3.4) then implies that

$$(3.6) \quad h^0(\mathcal{E}|_{L^2}(-2)) - h^1(\mathcal{E}|_{L^2}(-2)) = 3 - c_2(\mathcal{E}|_{L^2})$$

for any plane  $L^2$  in  $\mathbb{P}^n$ .

We claim here that

$$c_2(\mathcal{E}) \geq 3$$

on  $\mathbb{P}^n$ . Suppose, to the contrary, that  $c_2(\mathcal{E}) \leq 2$ . Then  $\chi(\mathcal{E}|_{L^2}(-2)) = 3 - c_2(\mathcal{E}|_{L^2}) \geq 1$  since  $c_2(\mathcal{E}) = c_2(\mathcal{E}|_{L^2})$  for any plane  $L^2$  in  $\mathbb{P}^n$ . Hence  $h^0(\mathcal{E}|_{L^2}(-2)) \neq 0$ . As we have seen, this implies that  $\mathcal{E}|_{L^2}$  lies either in the case (1), (2), or (5) of Theorem 1.1. Note here that if  $\mathcal{E}|_{L^2}$  lies in the case (5) then  $c_2(\mathcal{E}|_{L^2}) = 3$ . This contradicts that  $c_2(\mathcal{E}) \leq 2$ . Thus  $\mathcal{E}|_{L^2}$  actually lies either in the case (1) or (2) of Theorem 1.1. In particular it is isomorphic to a direct sum of line bundles. Hence  $\mathcal{E}$  is also isomorphic to a direct sum of line bundles by the splitting criterion [OSS80, Theorem 2.3.2] of Horrocks. Since  $h^0(\mathcal{E}|_{L^2}(-2)) \neq 0$ , this implies that  $h^0(\mathcal{E}(-2)) \neq 0$ , which contradicts the assumption (3.1). Hence the claim follows.

Since  $\mathcal{E}|_{L^2}$  is nef, we have  $0 \leq H(\mathcal{E}|_{L^2})^{r+1} = c_1(\mathcal{E}|_{L^2})^2 - c_2(\mathcal{E}|_{L^2})$ , i.e.,  $c_2(\mathcal{E}|_{L^2}) \leq 9$ . Hence we have

$$c_2(\mathcal{E}) \leq 9.$$

Recall here the well-known non-negativity (see, e.g., [Laz04, Theorem 8.2.1]) of the top Chern class of a nef vector bundle  $\mathcal{E}$  that

$$(3.7) \quad c_n(\mathcal{E}) \geq 0.$$

We shall divide the proof of Theorem 1.1 according to the value of  $c_2(\mathcal{E})$ , where  $3 \leq c_2(\mathcal{E}) \leq 7$ .

Note that  $H(\mathcal{E}|_{L^2})$  is nef and big since  $c_2(\mathcal{E}) < 9$ . The vanishing (3.3) then implies that

$$(3.8) \quad H^q(\mathcal{E}|_{L^2}) = 0$$

for any  $q > 0$  and any plane  $L^2$  in  $\mathbb{P}^n$ . Together with the vanishing (3.2), this implies that

$$(3.9) \quad H^q(\mathcal{E}|_{L^3}(-1)) = 0$$

for any  $q \geq 2$  and any three-dimensional linear subspace  $L^3 \subset \mathbb{P}^n$ .

**3.1. Set-up for the two-dimensional case.** In this subsection, we assume that  $n = 2$ . In this case, it follows from (3.1) and (3.6) that

$$(3.10) \quad h^1(\mathcal{E}(-2)) = c_2(\mathcal{E}) - 3.$$

The Riemann-Roch formula (3.5) shows  $\chi(\mathcal{E}(-1)) = 6 - c_2(\mathcal{E})$  and  $\chi(\mathcal{E}) = 9 + r - c_2(\mathcal{E})$ . Since we have the vanishings (3.4) and (3.8), these formulas imply

$$(3.11) \quad h^0(\mathcal{E}(-1)) - h^1(\mathcal{E}(-1)) = 6 - c_2(\mathcal{E}),$$

$$(3.12) \quad h^0(\mathcal{E}) = 9 + r - c_2(\mathcal{E}).$$

Since we have an exact sequence  $0 \rightarrow \mathcal{E}(-2) \rightarrow \mathcal{E}(-1) \rightarrow \mathcal{E}|_{L^1}(-1) \rightarrow 0$ , we have an exact sequence

$$0 \rightarrow H^0(\mathcal{E}(-1)) \rightarrow K^{\oplus 3} \rightarrow H^1(\mathcal{E}(-2)) \rightarrow H^1(\mathcal{E}(-1)) \rightarrow 0.$$

In particular we see that

$$(3.13) \quad h^1(\mathcal{E}(-2)) \geq h^1(\mathcal{E}(-1)).$$

Throughout the case where  $n = 2$ , we apply to  $\mathcal{E}$  the Bondal spectral sequence (2.1)

$$(3.14) \quad E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

It is clear that  $E_2^{p,q} = 0$  if  $q < 0$  or  $p > 0$ . The vanishing (3.4) shows that  $E_2^{p,q} = 0$  if  $q \geq 2$ . Since  $H^1(\mathcal{E}) = 0$  by (3.8), the right  $A$ -module  $\text{Ext}^1(G, \mathcal{E})$  fits in an exact sequence

$$0 \rightarrow S_1^{\oplus h^1(\mathcal{E}(-1))} \rightarrow \text{Ext}^1(G, \mathcal{E}) \rightarrow S_2^{\oplus h^1(\mathcal{E}(-2))} \rightarrow 0.$$

Since  $S_1 \otimes_A^L G \cong \Omega_{\mathbb{P}^2}(1)[1]$  and  $S_2 \otimes_A^L G \cong \mathcal{O}(-1)[2]$  by (2.4), the sequence above induces the following distinguished triangle in  $D^b(X)$ :

$$\mathcal{O}(-1)^{\oplus h^1(\mathcal{E}(-2))}[1] \rightarrow \Omega_{\mathbb{P}^2}(1)^{\oplus h^1(\mathcal{E}(-1))}[1] \rightarrow \text{Ext}^1(G, \mathcal{E}) \otimes_A^L G \rightarrow .$$

Since  $E_2^{p,1} = \mathcal{H}^p(\text{Ext}^1(G, \mathcal{E}) \otimes_A^L G)$ , the triangle above shows that  $E_2^{p,1} = 0$  unless  $p = -2$  or  $-1$  and that  $E_2^{-2,1}$  and  $E_2^{-1,1}$  fit in the following exact sequence of coherent sheaves:

$$(3.15) \quad 0 \rightarrow E_2^{-2,1} \rightarrow \mathcal{O}(-1)^{\oplus h^1(\mathcal{E}(-2))} \rightarrow \Omega_{\mathbb{P}^2}(1)^{\oplus h^1(\mathcal{E}(-1))} \rightarrow E_2^{-1,1} \rightarrow 0.$$

It follows from (3.1) that the right  $A$ -module  $\text{Hom}(G, \mathcal{E})$  has, as in (2.3), a projective resolution of the form

$$0 \rightarrow P_0^{\oplus 3e_{0,1}} \rightarrow P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}) \rightarrow 0$$

where  $e_{0,0} = h^0(\mathcal{E})$  and  $e_{0,1} = h^0(\mathcal{E}(-1))$ . In particular, we see that  $E_2^{p,0} = 0$  if  $p < -1$ . Now the Bondal spectral sequence implies that  $E_2^{-1,0} = 0$  and that  $\mathcal{E}$  fits in an exact sequence

$$(3.16) \quad 0 \rightarrow E_3^{0,0} \rightarrow \mathcal{E} \rightarrow E_2^{-1,1} \rightarrow 0.$$

Since  $E_2^{-1,0} = 0$ ,  $E_2^{0,0}$  fits in an exact sequence

$$(3.17) \quad 0 \rightarrow \mathcal{O}^{\oplus 3e_{0,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0.$$

Since  $\mathcal{E}$  is nef, the sequence (3.16) shows that, for the normalization  $C$  of any curve  $C'$  in  $\mathbb{P}^2$ , the pullback  $E_2^{-1,1}|_C$  of  $E_2^{-1,1}$  to  $C$  cannot admit a line bundle of negative degree as a quotient. The exact sequence (3.15) then indicates that there are some relations between  $h^1(\mathcal{E}(-2))$  and  $h^1(\mathcal{E}(-1))$ ; we shall explore these relations in Lemma 5.1.

**3.2. Set-up for the three-dimensional case.** The Riemann-Roch formula for a twisted vector bundle  $\mathcal{E}(t)$  on  $\mathbb{P}^3$  is

$$\chi(\mathcal{E}(t)) = \frac{1}{6} \{ c_1^3 - 3c_1c_2 + 3c_3 + 3(c_1^2 - 2c_2)(t+2) + c_1(3t^2 + 12t + 11) + r(t+3)(t+2)(t+1) \}$$

where  $c_i = c_i(\mathcal{E})$  for  $1 \leq i \leq 3$ . Since  $c_1 = 3$ , the formula above is reduced to the following:

$$\chi(\mathcal{E}(t)) = \frac{1}{2} \{ 9 - 3c_2 + c_3 + (9 - 2c_2)(t+2) + (3t^2 + 12t + 11) \} + \frac{r}{6} (t+3)(t+2)(t+1).$$

In particular, we have

$$(3.18) \quad \chi(\mathcal{E}(-3)) = 1 - \frac{1}{2}(c_2 - c_3),$$

$$(3.19) \quad \chi(\mathcal{E}(-2)) = 4 - \frac{1}{2}(3c_2 - c_3),$$

$$(3.20) \quad \chi(\mathcal{E}(-1)) = 10 - \frac{1}{2}(5c_2 - c_3).$$

By taking into account the vanishing (3.2), we also have

$$(3.21) \quad h^0(\mathcal{E}) = 19 - \frac{1}{2}(7c_2 - c_3) + r,$$

$$(3.22) \quad h^0(\mathcal{E}(1)) = 31 - \frac{1}{2}(9c_2 - c_3) + 4r.$$

For a nef vector bundle  $\mathcal{E}$  of rank  $r$  on  $\mathbb{P}^3$ , we have  $0 \leq H(\mathcal{E})^{r+2} = c_3 - 2c_1c_2 + c_1^3$ . Since  $c_1 = 3$ , we have

$$(3.23) \quad c_3 \geq 6c_2 - 27.$$

Inequality (3.23) gives a more strong condition on  $c_3$  than  $c_3 \geq 0$  if  $c_2 \geq 5$ .

**3.3. Set-up for the four-dimensional case.** The Riemann-Roch formula for a twisted vector bundle  $\mathcal{E}(t)$  on  $\mathbb{P}^4$  is

$$\begin{aligned} \chi(\mathcal{E}(t)) = \frac{1}{24} \{ & c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4 + 2(c_1^3 - 3c_1c_2 + 3c_3)(2t + 5) \\ & + (c_1^2 - 2c_2)(6t^2 + 30t + 35) + c_1(4t^3 + 30t^2 + 70t + 50) \\ & + r(t + 4)(t + 3)(t + 2)(t + 1) \} \end{aligned}$$

where  $c_i = c_i(\mathcal{E})$  for  $1 \leq i \leq 4$ . Since  $c_1 = 3$ , the formula above is reduced to the following:

$$\begin{aligned} \chi(\mathcal{E}(t)) = & \frac{27}{8} - \frac{3}{2}c_2 + \frac{c_3}{2} + \frac{c_2^2}{12} - \frac{c_4}{6} + \frac{1}{4}(9 - 3c_2 + c_3)(2t + 5) \\ & + \frac{1}{24}(9 - 2c_2)(6t^2 + 30t + 35) + \frac{1}{4}(2t^3 + 15t^2 + 35t + 25) \\ & + \frac{1}{24}r(t + 4)(t + 3)(t + 2)(t + 1). \end{aligned}$$

In particular, combining this with (3.2), we have

$$(3.24) \quad h^0(\mathcal{E}(-1)) = 15 - \frac{14}{3}c_2 + \frac{5}{4}c_3 + \frac{c_2^2}{12} - \frac{c_4}{6}.$$

For a nef vector bundle  $\mathcal{E}$  of rank  $r$  on  $\mathbb{P}^4$ , we have  $0 \leq H(\mathcal{E})^{r+3} = c_1^4 - 3c_1^2c_2 + 2c_1c_3 + c_2^2 - c_4$ . Since  $c_1 = 3$ , we have

$$0 \leq H(\mathcal{E})^{r+3} = 81 - 27c_2 + 6c_3 + c_2^2 - c_4$$

and thus

$$(3.25) \quad c_4 \leq 81 - 27c_2 + 6c_3 + c_2^2.$$

#### 4. LEMMAS

In this section, we collect lemmas applied several times in the proof of Theorem 1.1.

**Lemma 4.1.** *Let  $s$  be any non-zero element of  $H^0(T_{\mathbb{P}^n}(-1))$ . Then the zero locus  $(s)_0$  of  $s$  is a (reduced) point.*

*Proof.* Let  $(x_0, \dots, x_n)$  be a system of homogeneous coordinates of  $\mathbb{P}^n$ . We have an exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}^n}(1) \xrightarrow{\varphi} \mathcal{O}^{\oplus n+1} \xrightarrow{\psi} \mathcal{O}(1) \rightarrow 0$$



where  $(e_0, \dots, e_n)$  is the free basis of  $\mathcal{O}^{\oplus n+1}$ ,  $\varphi(x_j d(x_i/x_j)) = e_i - (x_i/x_j)e_j$ , and  $\psi$  is the evaluation map:  $\psi(e_i) = x_i$ . Dualizing the sequence above, we get an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus n+1} \rightarrow T_{\mathbb{P}^n}(-1) \rightarrow 0.$$

Let  $(a_0, \dots, a_n)$  be an element of  $H^0(\mathcal{O}^{\oplus n+1})$  corresponding to  $s$  via the isomorphism  $H^0(\mathcal{O}^{\oplus n+1}) \cong H^0(T_{\mathbb{P}^n}(-1))$ . By composing  $\varphi$  with the morphism  $\mathcal{O}^{\oplus n+1} \rightarrow \mathcal{O}$  sending  $e_i$  to  $a_i$ , we infer that the ideal sheaf  $\mathcal{I}$  of the zero locus  $(s)_0$  of  $s$  is generated by homogeneous polynomials  $a_i x_j - a_j x_i$  ( $0 \leq i, j \leq n$ ). Hence  $\mathcal{I}$  is the ideal sheaf of the point  $(a_0, \dots, a_n)$ . Therefore  $(s)_0$  is a (reduced) point.  $\square$

**Lemma 4.2.** *Let  $s$  be a non-zero element of  $H^0(\Omega_{\mathbb{P}^n}(2))$ . Then the zero locus  $(s)_0$  of  $s$  is a (possibly empty) linear subspace of codimension  $c \geq 2$  with  $c$  even. In particular, if  $n = 3$ , then  $(s)_0$  is either empty or a line.*

*Proof.* Consider the following exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}^n}(2) \rightarrow \mathcal{O}(1)^{\oplus n+1} \xrightarrow{\psi(1)} \mathcal{O}(2) \rightarrow 0.$$

Since  $H^0(\Omega_{\mathbb{P}^n}(1)) = 0$ ,  $(s)_0$  has codimension  $c \geq 2$ . Let  $\bar{s}$  be the image of  $s$  by the map  $H^0(\Omega_{\mathbb{P}^n}(2)) \rightarrow H^0(\mathcal{O}(1)^{\oplus n+1})$ . Since  $\Omega_{\mathbb{P}^n}(2) \rightarrow \mathcal{O}(1)^{\oplus n+1}$  is a subbundle morphism,  $(s)_0 = (\bar{s})_0$  as closed subschemes. Note here that  $\bar{s}$  can be regarded as a row vector  $(l_0, l_1, \dots, l_n)$  of linear form entries and that  $(\bar{s})_0$  is the common zeros of  $l_0 = 0, \dots, l_n = 0$ ; thus  $(\bar{s})_0$  is a (possibly empty) linear subspace of  $\mathbb{P}^n$ . Let  $(x_0, \dots, x_n)$  be a system of homogeneous coordinates of  $\mathbb{P}^n$ . Then  $l_j$  can be written as  $l_j = \sum_{i=0}^n a_{ij} x_i$  for some  $a_{ij} \in K$ . Since  $\bar{s} \in \text{Ker } \psi(1)$ ,  $\sum_{j=0}^n l_j x_j = 0$ . This implies that the matrix  $A$  is skew-symmetric where  $A = [a_{ij}]$ . Since the characteristic of  $K$  is not two, the rank of  $A$  is even. Therefore  $c$  is even.  $\square$

**Lemma 4.3.** *Let  $\mathcal{F}$  be a coherent sheaf on  $\mathbb{P}^n$ . Suppose that there exists an exact sequence*

$$0 \rightarrow \mathcal{O}^{\oplus a} \rightarrow \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c} \rightarrow \mathcal{F} \rightarrow 0$$

*where  $a, b$ , and  $c$  are non-negative integers. If  $\mathcal{F}$  is a vector bundle and if  $b \leq n$ , then  $\mathcal{F} \cong \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c-a}$ .*

*Proof.* Consider the composite  $\mathcal{O}^{\oplus a} \rightarrow \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c} \rightarrow \mathcal{O}^{\oplus c}$  of the injection  $\mathcal{O}^{\oplus a} \rightarrow \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c}$  and the projection  $\mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c} \rightarrow \mathcal{O}^{\oplus c}$ . We claim first that this composite is injective. Suppose, to the contrary, that there exists a subbundle  $\mathcal{O} \rightarrow \mathcal{O}^{\oplus a}$  such that the restriction of the composite  $\mathcal{O}^{\oplus a} \rightarrow \mathcal{O}^{\oplus c}$  to the subbundle  $\mathcal{O}$  is zero. Then the morphism  $\mathcal{O}^{\oplus a} \rightarrow \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c}$  induces a morphism  $\mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus b}$ . Since  $\mathcal{F}$  is a vector bundle, the injection  $\mathcal{O}^{\oplus a} \rightarrow \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c}$  is a subbundle morphism. Hence so is  $\mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus b}$ . On the other hand, since  $b \leq n$ , any (at most)  $b$  hyperplanes in  $\mathbb{P}^n$  have common zeros, which implies that any morphism  $\mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus b}$  cannot be a subbundle morphism. This is a contradiction. Hence the claim holds. Since the composite  $\mathcal{O}^{\oplus a} \rightarrow \mathcal{O}^{\oplus c}$  comes from a map of vector spaces, this implies that  $\mathcal{O}^{\oplus a}$  can be regarded as a direct summand of  $\mathcal{O}^{\oplus c}$ . Hence  $\mathcal{F} \cong \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c-a}$ .  $\square$

The following lemma can be seen as an refinement of [Ohn14, Proposition 2.7].

**Lemma 4.4.** *Let  $\mathcal{E}$  be a nef vector bundle on  $\mathbb{P}^n$ . Suppose that  $\mathcal{E}$  fits in an exact sequence of the form*

$$0 \rightarrow \mathcal{F} \rightarrow \bigoplus_{j=0}^{n-1} \mathcal{O}(j)^{\oplus e_{1,j}} \rightarrow \bigoplus_{j=1}^n \mathcal{O}(j)^{\oplus e_{0,j}} \rightarrow \mathcal{E}(2) \rightarrow 0$$

for some coherent sheaf  $\mathcal{F}$  on  $\mathbb{P}^n$ . If  $e_{1,0} < n$ , then  $e_{1,1} \geq e_{0,1}$  and the sequence above can be reduced to the following exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{F}(-1) \rightarrow \bigoplus_{j=1}^{n-2} \mathcal{O}(j)^{\oplus e_{1,j+1}} \oplus \mathcal{O}^{\oplus e_{1,1}-e_{0,1}} \oplus \mathcal{O}(-1)^{\oplus e_{1,0}} \\ \rightarrow \bigoplus_{j=1}^{n-1} \mathcal{O}(j)^{\oplus e_{0,j+1}} \rightarrow \mathcal{E}(1) \rightarrow 0. \end{aligned}$$

*Proof.* Consider the composite

$$\mathcal{O}(1)^{\oplus e_{1,1}} \rightarrow \bigoplus_{j=0}^{n-1} \mathcal{O}(j)^{\oplus e_{1,j}} \rightarrow \bigoplus_{j=1}^n \mathcal{O}(j)^{\oplus e_{0,j}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}}$$

of the inclusion  $\mathcal{O}(1)^{\oplus e_{1,1}} \rightarrow \bigoplus_{j=0}^{n-1} \mathcal{O}(j)^{\oplus e_{1,j}}$ , the morphism  $\bigoplus_{j=0}^{n-1} \mathcal{O}(j)^{\oplus e_{1,j}} \rightarrow \bigoplus_{j=1}^n \mathcal{O}(j)^{\oplus e_{0,j}}$ , and the projection  $\bigoplus_{j=1}^n \mathcal{O}(j)^{\oplus e_{0,j}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}}$ . We claim that this composite is surjective. Assume, to the contrary, that it is not surjective. Then there exists a surjection  $\mathcal{O}(1)^{\oplus e_{0,1}} \rightarrow \mathcal{O}(1)$  such that the composite  $\mathcal{O}(1)^{\oplus e_{1,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \rightarrow \mathcal{O}(1)$  is zero. This implies that the composite  $\bigoplus_{j=0}^{n-1} \mathcal{O}(j)^{\oplus e_{1,j}} \rightarrow \bigoplus_{j=1}^n \mathcal{O}(j)^{\oplus e_{0,j}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \rightarrow \mathcal{O}(1)$  induces a morphism

$$\mathcal{O}^{\oplus e_{1,0}} \cong (\bigoplus_{j=0}^{n-1} \mathcal{O}(j)^{\oplus e_{1,j}}) / (\bigoplus_{j=1}^n \mathcal{O}(j)^{\oplus e_{1,j}}) \rightarrow \mathcal{O}(1),$$

whose quotient is  $\mathcal{O}_L(1)$  where  $L$  is a linear subspace of dimension at least  $n - e_{1,0} \geq 1$  in  $\mathbb{P}^n$ . Hence  $\mathcal{E}(2)$  has  $\mathcal{O}_L(1)$  as a quotient, and thus  $\mathcal{E}|_L$  has  $\mathcal{O}_L(-1)$  as a quotient, which contradicts that  $\mathcal{E}$  is nef. Therefore the claim holds, and the sequence above can be reduced to the desired exact sequence.  $\square$

## 5. A KEY LEMMA FOR THE TWO DIMENSIONAL CASE

The following lemma together with the exact sequences (3.15) and (3.16) plays a crucial role in the proof of Theorem 1.1.

**Lemma 5.1.** *Let*

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}(-1)^{\oplus a} \xrightarrow{\mu} \Omega_{\mathbb{P}^2}(1)^{\oplus b} \rightarrow \mathcal{Q} \rightarrow 0$$

*be an exact sequence of coherent sheaves on  $\mathbb{P}^2$ , where  $a$  and  $b$  are non-negative integers. Suppose that, for the normalization  $C$  of any curve  $C'$  in  $\mathbb{P}^2$ , the pullback  $\mathcal{Q}|_C$  of  $\mathcal{Q}$  to  $C$  does not admit a line bundle of negative degree as a quotient.*

(1) *If  $b \geq 1$ , consider a surjection  $p : \Omega_{\mathbb{P}^2}(1)^{\oplus b} \rightarrow \Omega_{\mathbb{P}^2}(1)$ , and let*

$$\varphi : \mathcal{O}(-1)^{\oplus a} \rightarrow \Omega_{\mathbb{P}^2}(1)$$

*be the composite of  $\mu$  and  $p$ . Then  $H^0(\varphi(1)) : H^0(\mathcal{O}^{\oplus a}) \rightarrow H^0(\Omega_{\mathbb{P}^2}(2))$  is surjective; consequently  $a \geq 3$  and  $\varphi$  is surjective. Moreover we have an exact sequence*

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-3} \xrightarrow{\mu_1} \Omega_{\mathbb{P}^2}(1)^{\oplus b-1} \rightarrow \mathcal{Q} \rightarrow 0.$$

(2) *If  $b \geq 2$ , consider a surjection  $q : \Omega_{\mathbb{P}^2}(1)^{\oplus b-1} \rightarrow \Omega_{\mathbb{P}^2}(1)$ , and let*

$$\varphi_1 : \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-3} \rightarrow \Omega_{\mathbb{P}^2}(1)$$

*be the composite of  $\mu_1$  and  $q$ . Then the image of  $H^0(\varphi_1(1))$  has dimension two or three; consequently  $a \geq 5$ .*

(a) *If  $\dim \operatorname{Im} H^0(\varphi_1(1)) = 2$ , then we have the following exact sequence*

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1)^{\oplus a-5} \xrightarrow{\nu_2} \Omega_{\mathbb{P}^2}(1)^{\oplus b-2} \rightarrow \mathcal{Q} \rightarrow k(w) \rightarrow 0$$

*for some point  $w$  in  $\mathbb{P}^2$ , where  $k(w)$  denotes the residue field of  $w$ .*

(b) *If  $\dim \operatorname{Im} H^0(\varphi_1(1)) = 3$ , then  $a \geq 6$  and we have the following exact sequence*

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}(-2)^{\oplus 2} \oplus \mathcal{O}(-1)^{\oplus a-6} \xrightarrow{\mu_2} \Omega_{\mathbb{P}^2}(1)^{\oplus b-2} \rightarrow \mathcal{Q} \rightarrow 0.$$

*Proof.* (1) First note that there exists a surjection  $\mathcal{Q} \rightarrow \text{Coker } \varphi$ .

If  $H^0(\varphi(1)) = 0$ , then  $\varphi = 0$  and  $\text{Coker } \varphi \cong \Omega_{\mathbb{P}^2}(1)$ . Since  $\Omega_{\mathbb{P}^2}|_{L^1}(1) \cong \mathcal{O}_{L^1} \oplus \mathcal{O}_{L^1}(-1)$  for any line  $L^1$  in  $\mathbb{P}^2$ , this implies that  $\mathcal{Q}|_{L^1}$  has  $\mathcal{O}_{L^1}(-1)$  as a quotient, which contradicts the assumption on  $\mathcal{Q}$ . Therefore  $H^0(\varphi(1)) \neq 0$ .

Suppose that the image of  $H^0(\varphi(1))$  has dimension one, and let  $s$  be a non-zero element in the image in  $H^0(\Omega_{\mathbb{P}^2}(2)) \cong H^0(T_{\mathbb{P}^2}(-1))$ . Then the zero locus of  $s$  is a (reduced) point  $z$  by Lemma 4.1 and  $\text{Coker } \varphi$  is isomorphic to the ideal sheaf  $\mathcal{I}_z$  of  $z$ . If we restrict the surjection  $\mathcal{Q} \rightarrow \mathcal{I}_z$  to a line  $L^1$  passing through  $z$ , we infer that  $\mathcal{Q}|_{L^1}$  has  $\mathcal{O}_{L^1}(-z)$  as a quotient, which contradicts the assumption on  $\mathcal{Q}$ . Therefore the image of  $H^0(\varphi(1))$  has dimension  $\geq 2$ .

Suppose that the image of  $H^0(\varphi(1))$  has dimension two, and let  $(s, t)$  be a basis of the image. As above, let  $z$  be the zero locus of  $s$  and  $\mathcal{I}_z$  the ideal sheaf of  $z$ . Then  $t$  induces an injection  $\mathcal{O} \rightarrow \mathcal{I}_z(1)$ , and this gives a line  $L^1$  passing through  $z$ . The quotient of the  $\mathcal{O}(-1)$ -twist of this injection is  $\mathcal{O}_{L^1}(-z)$  and is isomorphic to  $\text{Coker } \varphi$ . Thus  $\mathcal{Q}|_{L^1}$  has  $\mathcal{O}_{L^1}(-z)$  as a quotient, which contradicts the assumption on  $\mathcal{Q}$ . Therefore the image of  $H^0(\varphi(1))$  has dimension  $\geq 3$ , i.e.,  $H^0(\varphi(1))$  is surjective. Hence  $\varphi$  is surjective and  $\text{Ker } \varphi \cong \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-3}$ . Now  $\text{Ker } p \cong \Omega_{\mathbb{P}^2}(1)^{\oplus b-1}$ , and the morphism  $\text{Ker } \varphi \rightarrow \text{Ker } p$  induced by  $\mu$  extends to the exact sequence in the statement.

(2) Note first that there exists a surjection  $\mathcal{Q} \rightarrow \text{Coker } \varphi_1$ . Denote by  $\iota$  the inclusion  $\mathcal{O}(-1)^{\oplus a-3} \rightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-3}$ ; consider the composite  $\varphi_1 \circ \iota : \mathcal{O}(-1)^{\oplus a-3} \rightarrow \Omega_{\mathbb{P}^2}(1)$  of  $\iota$  and  $\varphi_1$ . We see that  $\varphi_1$  induces a morphism  $\bar{\varphi}_1 : \text{Coker}(\iota) \rightarrow \text{Coker}(\varphi_1 \circ \iota)$  and that  $\text{Coker}(\iota) \cong \mathcal{O}(-2)$ . Moreover we have the following long exact sequence

$$(5.1) \quad 0 \rightarrow \text{Ker}(\varphi_1 \circ \iota) \rightarrow \text{Ker } \varphi_1 \rightarrow \mathcal{O}(-2) \xrightarrow{\bar{\varphi}_1} \text{Coker}(\varphi_1 \circ \iota) \rightarrow \text{Coker } \varphi_1 \rightarrow 0$$

by the snake lemma. Therefore  $\text{Coker } \varphi_1 \cong \text{Coker } \bar{\varphi}_1$ .

If  $H^0(\varphi_1(1)) = 0$ , then  $\varphi_1 \circ \iota = 0$  and thus  $\text{Coker}(\varphi_1 \circ \iota) \cong \Omega_{\mathbb{P}^2}(1)$ . If  $\bar{\varphi}_1 = 0$ , then  $\text{Coker } \varphi_1 \cong \text{Coker}(\varphi_1 \circ \iota)$ . Hence  $\mathcal{Q}$  has  $\Omega_{\mathbb{P}^2}(1)$  as a quotient, which contradicts the assumption on  $\mathcal{Q}$ . Therefore  $\bar{\varphi}_1 : \mathcal{O}(-2) \rightarrow \Omega_{\mathbb{P}^2}(1)$  is a non-zero morphism. Let  $\bar{s}$  be the non-zero section of  $H^0(\Omega_{\mathbb{P}^2}(3))$  determined by  $\bar{\varphi}_1$ . Suppose that  $\bar{s}$  can be decomposed as  $\bar{s} = sl$ , where  $s$  is a non-zero section of  $H^0(\Omega_{\mathbb{P}^2}(2)) \cong H^0(T_{\mathbb{P}^2}(-1))$  and  $l$  is a non-zero section of  $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ . Then  $\bar{\varphi}_1$  is the composite of the inclusion  $\mathcal{O}(-2) \rightarrow \mathcal{O}(-1)$  determined by  $l$  and the morphism  $\mathcal{O}(-1) \rightarrow \Omega_{\mathbb{P}^2}(1)$  determined by  $s$ . The cokernel of the inclusion  $\mathcal{O}(-2) \rightarrow \mathcal{O}(-1)$  is isomorphic to  $\mathcal{O}_{L^1}(-1)$ , where  $L^1$  is the line defined by  $l$ , and the cokernel of the morphism  $\mathcal{O}(-1) \rightarrow \Omega_{\mathbb{P}^2}(1)$  is isomorphic to  $\mathcal{I}_z$ , where  $\mathcal{I}_z$  is the ideal sheaf of the zero (reduced) point  $z$  of  $s$ , as in the proof of the case (1); thus  $\text{Coker } \bar{\varphi}_1$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}_{L^1}(-1) \rightarrow \text{Coker } \bar{\varphi}_1 \rightarrow \mathcal{I}_z \rightarrow 0.$$

Therefore  $\text{Coker } \varphi_1$  (and hence  $\mathcal{Q}$ ) has  $\mathcal{I}_z$  as a quotient, which contradicts the assumption on  $\mathcal{Q}$  as we have seen in the proof of (1). Hence  $\bar{s}$  does not factor as  $\bar{s} = sl$ . Consequently  $\bar{s}$  is a regular section of  $\Omega_{\mathbb{P}^2}(3) \cong T_{\mathbb{P}^2}$ , i.e., the zero locus  $Z$  of  $\bar{s}$  has dimension  $\leq 0$ . Since  $c_2(T_{\mathbb{P}^2}) = 3$ ,  $Z$  is thereby a 0-dimensional closed subscheme of length three. Note that  $\text{Coker } \bar{\varphi}_1 \cong \mathcal{I}_Z(1)$ . Now take a line  $L^1$  that intersects  $Z$  in length at least two. Then  $\mathcal{I}_Z(1)|_{L^1}$  is a line bundle of negative degree, which contradicts the assumption on  $\mathcal{Q}$ . Therefore  $H^0(\varphi_1(1)) \neq 0$ .

Suppose that the image of  $H^0(\varphi_1(1))$  has dimension one. Then  $\text{Coker}(\varphi_1 \circ \iota)$  is isomorphic to the ideal sheaf  $\mathcal{I}_z$  of some point  $z$ , as is shown in the proof of (1). If  $\bar{\varphi}_1 = 0$ ,

then  $\text{Coker } \bar{\varphi}_1 \cong \mathcal{I}_z$ , which contradicts the assumption on  $\mathcal{Q}$  as we have seen in the proof of (1). Hence  $\bar{\varphi}_1 : \mathcal{O}(-2) \rightarrow \mathcal{I}_z$  is non-zero. Then  $\text{Coker } \bar{\varphi}_1$  is isomorphic to  $\mathcal{O}_C(-z)$  for some conic  $C$  on  $\mathbb{P}^2$ , which contradicts the assumption on  $\mathcal{Q}$ . Therefore the image of  $H^0(\varphi_1(1))$  has dimension at least two.

(2) (a) Suppose that the image of  $H^0(\varphi_1(1))$  has dimension two. Then  $\text{Coker}(\varphi_1 \circ \iota) \cong \mathcal{O}_{L^1}(-z)$  for some line  $L^1$  in  $\mathbb{P}^2$  and some point  $z$  in  $L^1$ , as is shown in the proof of (1). If  $\bar{\varphi}_1 = 0$ , then  $\text{Coker } \bar{\varphi}_1 \cong \mathcal{O}_{L^1}(-z)$ , which contradicts the assumption on  $\mathcal{Q}$ . Hence  $\bar{\varphi}_1 : \mathcal{O}(-2) \rightarrow \mathcal{O}_{L^1}(-z)$  is non-zero. Then  $\text{Coker } \bar{\varphi}_1 \cong k(w)$  for some point  $w$  on  $\mathbb{P}^2$ , and  $\text{Ker } \bar{\varphi}_1 \cong \mathcal{O}(-3)$ . The long exact sequence (5.1) induces an exact sequence

$$0 \rightarrow \text{Ker}(\varphi_1 \circ \iota) \rightarrow \text{Ker } \varphi_1 \rightarrow \text{Ker } \bar{\varphi}_1 \rightarrow 0.$$

Note here that  $\text{Ker } \varphi_1 \circ \iota \cong \mathcal{O}(-1)^{\oplus a-5}$ ; the above sequence then implies that  $\text{Ker } \varphi_1 \cong \mathcal{O}(-3) \oplus \mathcal{O}(-1)^{\oplus a-5}$ . Now we get the exact sequence in the statement by the snake lemma.

(2) (b) Suppose that the image of  $H^0(\varphi_1(1))$  has dimension  $\geq 3$ . Then  $H^0(\varphi_1(1))$  is surjective, and thus  $\varphi_1 \circ \iota$  is surjective. Hence  $\text{Ker } \varphi_1 \circ \iota \cong \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-6}$ . Then the long exact sequence (5.1) gives an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-6} \rightarrow \text{Ker } \varphi_1 \rightarrow \mathcal{O}(-2) \rightarrow 0.$$

Therefore  $\text{Ker } \varphi_1 \cong \mathcal{O}(-2)^{\oplus 2} \oplus \mathcal{O}(-1)^{\oplus a-6}$ . Now  $\text{Ker } q \cong \Omega_{\mathbb{P}^2}(1)^{\oplus b-2}$ , and  $\mu_1$  induces a morphism  $\text{Ker } \varphi_1 \rightarrow \text{Ker } q$ , which extends to the exact sequence in the statement.  $\square$

## 6. THE CASE WHERE $c_2(\mathcal{E}) = 3$

In this section, we give a proof of Theorem 1.1 in case  $c_2(\mathcal{E}) = 3$ ; throughout this section, we assume that  $c_2(\mathcal{E}) = 3$ .

**6.1. The case where  $n = 2$ .** Suppose that  $n = 2$ . Since  $c_2(\mathcal{E}) = 3$ , it follows from (3.10) and (3.13) that  $h^1(\mathcal{E}(-2)) = h^1(\mathcal{E}(-1)) = 0$ . We apply the Bondal spectral sequence (3.14). The exact sequence (3.15) then shows the vanishing of  $E_2^{-2,1}$  and  $E_2^{-1,1}$ ; consequently the exact sequence (3.16) implies that  $\mathcal{E} \cong E_3^{0,0} \cong E_2^{0,0}$ . We have  $h^0(\mathcal{E}) = r + 6$  by (3.12). Since  $h^1(\mathcal{E}(-1)) = 0$ , we also have  $h^0(\mathcal{E}(-1)) = 3$  by (3.11). Therefore the exact sequence (3.17) implies that  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 9} \rightarrow \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r+6} \rightarrow \mathcal{E} \rightarrow 0.$$

Now consider the composite of the inclusion  $\mathcal{O}^{\oplus 9} \rightarrow \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r+6}$  and the projection  $\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r+6} \rightarrow \mathcal{O}^{\oplus r+6}$ , and let  $\mathcal{O}^{\oplus k}$  be its kernel and  $\mathcal{O}^{\oplus q}$  its cokernel. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus k} \rightarrow \mathcal{O}(1)^{\oplus 3} \rightarrow \mathcal{E} \rightarrow \mathcal{O}^{\oplus q} \rightarrow 0$$

by the snake lemma. Note here that the map  $\mathcal{O}(1)^{\oplus 3} \rightarrow \mathcal{E}$  is nothing but the evaluation map  $\text{Hom}(\mathcal{O}(1), \mathcal{E}) \otimes \mathcal{O}(1) \rightarrow \mathcal{E}$ . It follows from the same argument as in the proof of [Ohn14, Proposition 5.8] that the image of this map has rank  $\geq 2$  because  $\text{hom}(\mathcal{O}(1), \mathcal{E}) \geq 2$ . Hence  $k = 1$  or  $0$ , and consequently  $q = r - 2$  or  $r - 3$  respectively. Note here that the image of the evaluation map is a vector bundle since it is also the kernel of the surjection  $\mathcal{E} \rightarrow \mathcal{O}^{\oplus q}$  of vector bundles. Hence if  $k = 1$  then the image of the evaluation map is the tangent bundle  $T_{\mathbb{P}^2}$  and we have an exact sequence

$$0 \rightarrow T_{\mathbb{P}^2} \rightarrow \mathcal{E} \rightarrow \mathcal{O}^{\oplus r-2} \rightarrow 0.$$

Hence  $\mathcal{E} \cong T_{\mathbb{P}^2} \oplus \mathcal{O}^{\oplus r-2}$ . This is the case (4) of Theorem 1.1. If  $k = 0$ , then we have an exact sequence

$$0 \rightarrow \mathcal{O}(1)^{\oplus 3} \rightarrow \mathcal{E} \rightarrow \mathcal{O}^{\oplus r-3} \rightarrow 0.$$

Therefore  $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r-3}$ . This is the case (3) of Theorem 1.1.

**6.2. The case where  $n = 3$ .** Suppose that  $n = 3$ . We shall apply to  $\mathcal{E}$  the Bondal spectral sequence (2.1):

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if } p+q = 0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

As we have seen in §6,  $H^1(\mathcal{E}|_{L^2}(-2))$  and  $H^1(\mathcal{E}|_{L^2}(-1))$  vanish for any plane  $L^2 \subset \mathbb{P}^3$ . Now we can summarize (3.2), (3.4), (3.8), and this vanishing as follows:

$$H^q(\mathcal{E}|_{L^2}(-k)) = 0 \text{ if } q \geq 1 \text{ and } k \leq 2.$$

Since  $H^q(\mathcal{E}(-k)) = 0$  if  $q \geq 1$  and  $k \leq 0$  by (3.2), the vanishing above implies that  $H^q(\mathcal{E}(-k)) = 0$  if  $q \geq 2$  and  $k \leq 3$ . Note here that  $h^0(\mathcal{E}(-3)) = 0$  by (3.1). Therefore Formula (3.18) implies

$$0 \geq -h^1(\mathcal{E}(-3)) = \frac{c_3 - 1}{2}.$$

Hence  $c_3$  is odd and  $c_3 \leq 1$ . On the other hand,  $c_3 \geq 0$  by (3.7). Therefore  $c_3 = 1$  and  $H^1(\mathcal{E}(-3)) = 0$ . The vanishing of  $H^1(\mathcal{E}|_{L^2}(-2))$  then implies that of  $H^1(\mathcal{E}(-2))$ . Hence  $H^1(\mathcal{E}(-1))$  vanishes since so does  $H^1(\mathcal{E}|_{L^2}(-1))$ . Summing up,  $H^q(\mathcal{E}(-k))$  vanishes if  $q > 0$  and  $k \leq 3$ . Thus  $E_2^{p,q} = 0$  if  $q > 0$ . The Bondal spectral sequence then implies that  $E_2^{p,0} = 0$  unless  $p = 0$  and that  $\mathcal{E} \cong E_2^{0,0} \cong \text{Hom}(G, \mathcal{E}) \otimes_A G$ . The right  $A$ -module  $\text{Hom}(G, \mathcal{E})$  has, as in (2.3), a projective resolution of the form

$$0 \rightarrow P_0^{\oplus 4e_{0,1}} \rightarrow P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}) \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E})$  and  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E})$ . We see that  $e_{0,0} = r + 9$  by (3.21). Since  $H^q(\mathcal{E}(-1)) = 0$  for  $q > 0$ , the Riemann-Roch formula (3.20) shows that  $e_{0,1} = 3$ . Since  $E_2^{p,0} = 0$  for  $p < 0$  and  $\mathcal{E} \cong \text{Hom}(G, \mathcal{E}) \otimes_A G$ , we obtain the following exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 12} \rightarrow \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r+9} \rightarrow \mathcal{E} \rightarrow 0.$$

Now Lemma 4.3 shows that  $\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r-3} \cong \mathcal{E}$ . This is the case (3) of Theorem 1.1.

**6.3. The case where  $n \geq 4$ .** Suppose that  $n \geq 4$ . In this case,  $\mathcal{E}|_{L^3} \cong \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r-3}$  for any three-dimensional linear subspace  $L^3$  in  $\mathbb{P}^n$  as we have shown in §6.2. Hence  $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r-3}$  by the splitting criterion [OSS80, Theorem 2.3.2] of Horrocks.

## 7. THE CASE WHERE $c_2(\mathcal{E}) = 4$

In this section, we give a proof of Theorem 1.1 in case  $c_2(\mathcal{E}) = 4$ ; throughout this section, we assume that  $c_2(\mathcal{E}) = 4$ .

**7.1. The case where  $n = 2$ .** Suppose that  $n = 2$ . Since  $c_2(\mathcal{E}) = 4$ , we obtain  $h^1(\mathcal{E}(-2)) = 1$  by (3.10). Now we apply the Bondal spectral sequence (3.14); It follows from Lemma 5.1 (1) together with exact sequences (3.15) and (3.16) that  $H^1(\mathcal{E}(-1)) = 0$ , that  $E_2^{-2,1} \cong \mathcal{O}(-1)$ , and that  $E_2^{-1,1} = 0$ . Hence  $E_3^{0,0} \cong \mathcal{E}$  by (3.16). Therefore  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow E_2^{0,0} \rightarrow \mathcal{E} \rightarrow 0.$$

Since  $\mathcal{E}$  and  $\mathcal{O}(-1)$  are vector bundles, so is  $E_2^{0,0}$ . The Riemann-Roch formulas (3.11) and (3.12), imply that  $h^0(\mathcal{E}(-1)) = 2$  and that  $h^0(\mathcal{E}) = 5 + r$ . It then follows from (3.17) that  $E_2^{0,0}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 6} \rightarrow \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus 5+r} \rightarrow E_2^{0,0} \rightarrow 0.$$

Since  $E_2^{0,0}$  is a vector bundle, Lemma 4.3 shows that  $E_2^{0,0} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-1}$ ; thus we get the case (6) of Theorem 1.1.

**7.2. The case where  $n = 3$ .** Suppose that  $n = 3$ . As we have seen in §7.1, since  $c_2(\mathcal{E}|_{L^2}) = c_2(\mathcal{E}) = 4$ ,  $\mathcal{E}|_{L^2}$  fits in an exact sequence

$$(7.1) \quad 0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-1} \rightarrow \mathcal{E}|_{L^2} \rightarrow 0$$

for any plane  $L^2 \subset \mathbb{P}^3$ . Hence  $H^q(\mathcal{E}|_{L^2}(-k)) = 0$  if  $q > 0$  and  $k \leq 1$ . Since  $H^q(\mathcal{E}(-1)) = 0$  for all  $q \geq 2$  by (3.9), this implies that

$$(7.2) \quad H^q(\mathcal{E}(-2)) = 0$$

for all  $q \geq 2$ . The Riemann-Roch formula (3.19) together with (3.1) then shows that

$$(7.3) \quad 0 \geq -h^1(\mathcal{E}(-2)) = \frac{c_3 - 4}{2}.$$

Hence  $c_3$  is even and  $c_3 \leq 4$ . Inequality (3.7) then implies that  $c_3 = 0, 2$ , or  $4$ . Since  $c_2 = 4$  and  $c_3 \geq 0$ ,  $H(\mathcal{E})$  is nef and big by (3.23); the vanishing (3.3), in particular, shows that

$$(7.4) \quad H^1(\mathcal{E}(-1)) = 0.$$

It then follows from (3.9) and (3.20) that

$$(7.5) \quad h^0(\mathcal{E}(-1)) = \frac{c_3}{2}.$$

**7.2.1. The case where  $c_3 = 0$ .** We shall show that the case where  $c_3 = 0$  does not happen. Suppose, to the contrary, that  $c_3 = 0$ . Then we obtain from (7.3) and (7.5)

$$(7.6) \quad h^1(\mathcal{E}(-2)) = 2,$$

$$(7.7) \quad h^0(\mathcal{E}(-1)) = 0.$$

We shall apply to  $\mathcal{E}(1)$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q = 0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

It follows from (7.2), (3.9), and (3.2) that  $\text{Ext}^q(G, \mathcal{E}(1)) = 0$  if  $q \geq 2$ ; thus  $E_2^{p,q} = 0$  for all  $p$  if  $q \geq 2$ . It follows from (7.6), (7.4), and (3.2) that  $\text{Ext}^1(G, \mathcal{E}(1)) \cong S_3^{\oplus 2}$ ; thus  $E_2^{p,1} = 0$  unless  $p = -3$  and  $E_2^{-3,1} \cong \mathcal{O}(-1)^{\oplus 2}$  since  $S_3 \otimes_{\mathcal{L}_A} G \cong \mathcal{O}(-1)[3]$  by (2.5). The Bondal spectral sequence then implies that  $E_2^{-1,0} \cong E_2^{-3,1}$  and that  $\mathcal{E}(1) \cong E_2^{0,0}$ . It follows from

(7.7) and (2.3) that the right  $A$ -module  $\text{Hom}(G, \mathcal{E}(1))$  has a projective resolution of the form

$$0 \rightarrow P_0^{\oplus e_{1,0}} \rightarrow P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}(1)) \rightarrow 0$$

where  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(1))$ ,  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(1))$ , and  $e_{1,0} = 4e_{0,1}$ . Since  $E_2^{-1,0} \cong \mathcal{O}(-1)^{\oplus 2}$ ,  $\mathcal{E}(1)$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow \mathcal{O}^{\oplus e_{1,0}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

We see that  $e_{0,1} = r + 5$  by (3.21). Since  $\mathcal{E}(1)$  does not admit  $\mathcal{O}$  as a quotient, it follows from [Ohn14, Proposition 2.7] that the sequence above can be reduced to the following form

$$0 \rightarrow \mathcal{O}(-2)^{\oplus 2} \rightarrow \mathcal{O}(-1)^{\oplus e_{1,0} - e_{0,0}} \rightarrow \mathcal{O}^{\oplus r+5} \rightarrow \mathcal{E} \rightarrow 0.$$

Hence  $e_{1,0} - e_{0,0} = 7$ . Now  $\mathcal{E}$  is globally generated, so that  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus r-3} \rightarrow \mathcal{E} \rightarrow \mathcal{E}_0 \rightarrow 0$$

where  $\mathcal{E}_0$  is a globally generated vector bundle of rank three. Since  $c_3(\mathcal{E}_0) = c_3 = 0$ ,  $\mathcal{E}_0$  fits in an exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow 0$$

where  $\mathcal{E}_1$  is a globally generated vector bundle of rank two. Here we adopt the proof in [AM13, §4] to show that this case does not happen. Let  $s$  be a general element of  $H^0(\mathcal{E}_1)$ , and let  $Z$  be the zero locus of  $s$ . Since  $H^0(\mathcal{E}_1(-1)) \cong H^0(\mathcal{E}(-1)) = 0$  by (7.7),  $Z$  has codimension two in  $\mathbb{P}^3$ . Hence  $Z$  is a smooth curve, and the ideal sheaf  $\mathcal{I}_Z$  of  $Z$  fits in an exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}_1 \rightarrow \mathcal{I}_Z(3) \rightarrow 0.$$

Since  $H^0(\mathcal{I}_Z(2)) \cong H^0(\mathcal{E}_1(-1)) = 0$ ,  $Z$  is not contained in a quadric surface. Moreover  $\mathcal{I}_Z$  is generated by cubics since  $\mathcal{E}_1$  is globally generated. Let  $Z_0$  be any connected component of  $Z$ . Then  $\mathcal{E}_1|_{Z_0} \cong \mathcal{N}_{Z_0/\mathbb{P}^3}^\vee(3)$  where  $\mathcal{N}_{Z_0/\mathbb{P}^3}$  is the normal bundle of  $Z_0$  in  $\mathbb{P}^3$  and  $\mathcal{N}_{Z_0/\mathbb{P}^3}^\vee$  its dual. Hence  $\det \mathcal{N}_{Z_0/\mathbb{P}^3} \cong \mathcal{O}_{Z_0}(3)$  and  $\omega_{Z_0} \cong \omega_{\mathbb{P}^3} \otimes \det \mathcal{N}_{Z_0/\mathbb{P}^3} \cong \mathcal{O}_{Z_0}(-1)$ . Therefore  $Z_0$  is a smooth conic in  $\mathbb{P}^3$ . Since  $Z$  is not contained in any quadric surface, this implies that  $Z$  contains at least two distinct smooth conics  $Z_0$  and  $Z_1$ . Let  $\langle Z_i \rangle$  denote the plane containing  $Z_i$  ( $i = 0, 1$ ), and let  $L$  be the line  $\langle Z_0 \rangle \cap \langle Z_1 \rangle$ . Then each  $Z_i$  intersects  $L$  in two points, so that  $L$  intersects  $Z_0 \cup Z_1$  in four points. Hence  $Z$  has a 4-secant line  $L$ , but this contradicts the fact that  $\mathcal{I}_Z$  is generated by cubics. Therefore the case  $c_3 = 0$  does not happen.

**7.2.2. The case where  $c_3 = 2$ .** Suppose that  $c_3 = 2$ . It follows from (7.3) and (7.5) respectively that

$$(7.8) \quad h^1(\mathcal{E}(-2)) = 1,$$

$$(7.9) \quad h^0(\mathcal{E}(-1)) = 1.$$

We shall apply to  $\mathcal{E}(1)$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q = 0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

It follows from (7.2), (3.9), and (3.2) that  $\text{Ext}^q(G, \mathcal{E}(1)) = 0$  if  $q \geq 2$ ; thus  $E_2^{p,q} = 0$  for all  $p$  if  $q \geq 2$ . It follows from (7.8), (7.4), and (3.2) that  $\text{Ext}^1(G, \mathcal{E}(1)) \cong S_3$ ; thus  $E_2^{p,1} = 0$  unless  $p = -3$  and  $E_2^{-3,1} \cong \mathcal{O}(-1)$  since  $S_3 \otimes_A^L G \cong \mathcal{O}(-1)[3]$  by (2.5). The Bondal

spectral sequence then implies that  $E_2^{-1,0} \cong E_2^{-3,1}$ , that  $E_2^{-2,0} = 0$ , and that  $\mathcal{E}(1) \cong E_2^{0,0}$ . It follows from (7.9) and (2.2) that the right  $A$ -module  $\text{Hom}(G, \mathcal{E}(1))$  has a projective resolution of the form

$$0 \rightarrow P_0^{\oplus 16} \rightarrow P_1^{\oplus 4} \oplus P_0^{\oplus e_{1,0}} \rightarrow P_2 \oplus P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}(1)) \rightarrow 0$$

where  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(1))$ ,  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(1))$ , and  $e_{1,0} = 4e_{0,1} + 10$ . Since  $E_2^{-1,0} \cong \mathcal{O}(-1)$  and  $E_2^{-2,0} = 0$ , we obtain the following:

$$\begin{aligned} \text{Coker}(\mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}^{\oplus e_{1,0}} \xrightarrow{\beta} \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}}) &\cong \mathcal{E}(1); \\ 0 \rightarrow \mathcal{O}^{\oplus 16} \rightarrow \text{Ker } \beta \rightarrow \mathcal{O}(-1) &\rightarrow 0. \end{aligned}$$

Since  $\text{Ext}^1(\mathcal{O}(-1), \mathcal{O}^{\oplus 16}) = 0$ ,  $\text{Ker } \beta \cong \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1)$ . Therefore  $\mathcal{E}(1)$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1) \rightarrow \mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}^{\oplus e_{1,0}} \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since  $\mathcal{E}(1)$  does not admit  $\mathcal{O}$  as a quotient, it follows from [Ohn14, Proposition 2.7] that the sequence above can be reduced to the following form

$$(7.10) \quad 0 \rightarrow \mathcal{O}(-1)^{\oplus 16} \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1)^{\oplus e_{1,0}-e_{0,0}} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{E} \rightarrow 0.$$

Hence  $e_{1,0} - e_{0,0} = 20$  and  $e_{0,0} = r + 6$ . We split the sequence above into two exact sequences:

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-1)^{\oplus 16} \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1)^{\oplus 20} &\rightarrow \mathcal{G} \rightarrow 0; \\ 0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+6} &\rightarrow \mathcal{E} \rightarrow 0. \end{aligned}$$

Note here that the composite  $\mathcal{O}(1) \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+6} \rightarrow \mathcal{E}$  is injective. Hence the composite  $\mathcal{G} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+6} \rightarrow \mathcal{O}^{\oplus r+6}$  is injective. Moreover the composite  $\mathcal{O}^{\oplus 4} \rightarrow \mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1)^{\oplus 20} \rightarrow \mathcal{G}$  is injective. Therefore the composite  $\mathcal{O}^{\oplus 4} \rightarrow \mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1)^{\oplus 20} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+6} \rightarrow \mathcal{O}^{\oplus r+6}$  is injective. Hence the exact sequence (7.10) can be reduced to the following form

$$0 \rightarrow \mathcal{O}(-1)^{\oplus 16} \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 20} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{r+2} \rightarrow \mathcal{E} \rightarrow 0.$$

Since the composite  $\mathcal{O}(-1)^{\oplus 16} \rightarrow \mathcal{O}(-1)^{\oplus 16} \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 20}$  is injective, the sequence above can be reduced to the following form

$$(7.11) \quad 0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 4} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{r+2} \rightarrow \mathcal{E} \rightarrow 0.$$

This is the case (7) of Theorem 1.1.

**Remark 7.1.** *The exact sequence (7.11) induces the following exact sequence*

$$0 \rightarrow T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+2} \rightarrow \mathcal{E} \rightarrow 0,$$

*and, dualizing this sequence, we obtain the following exact sequence*

$$0 \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r+2} \rightarrow \Omega_{\mathbb{P}^3}(2) \rightarrow 0.$$

*Note that the injection  $H^0(\mathcal{E}^\vee) \rightarrow H^0(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r+2})$  induces a splitting injection  $\mathcal{O} \otimes H^0(\mathcal{E}^\vee) \rightarrow \mathcal{O}^{\oplus r+2}$  and that the composite of two splitting injection  $\mathcal{O} \otimes H^0(\mathcal{E}^\vee) \rightarrow \mathcal{O}^{\oplus r+2}$  and  $\mathcal{O}^{\oplus r+2} \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r+2}$  is equal to the composite of  $\mathcal{O} \otimes H^0(\mathcal{E}^\vee) \rightarrow \mathcal{E}^\vee$  and  $\mathcal{E}^\vee \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r+2}$ . Thus  $\mathcal{O} \otimes H^0(\mathcal{E}^\vee) \rightarrow \mathcal{E}^\vee$  is also a splitting injection. Hence  $\mathcal{E}^\vee \cong$*



$\mathcal{E}_0^\vee \oplus \mathcal{O} \otimes H^0(\mathcal{E}^\vee)$  for some vector bundle  $\mathcal{E}_0$  of rank  $s = r - h^0(\mathcal{E}^\vee)$ . Since  $c_3(\mathcal{E}_0) = 2 \neq 0$ , we infer that  $s \geq 3$ . Note that  $h^0(\mathcal{E}_0^\vee) = 0$  and that  $\mathcal{E}_0^\vee$  fits in an exact sequence

$$0 \rightarrow \mathcal{E}_0^\vee \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus s+2} \rightarrow \Omega_{\mathbb{P}^3}(2) \rightarrow 0.$$

Since  $h^0(\Omega_{\mathbb{P}^3}(2)) = 6$  by the Bott formula [OSS80, p. 8], we see that  $s \leq 4$ . Moreover  $h^1(\mathcal{E}_0^\vee) = 4 - s$ . The image of  $\mathcal{E}_0^\vee \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus s+2} \rightarrow \mathcal{O}(-1)$  is  $\mathcal{I}_Z(-1)$  for some closed subscheme  $Z$  in  $\mathbb{P}^3$ .

Suppose that  $Z = \emptyset$ . Let  $\mathcal{F}^\vee$  be the kernel of the surjection  $\mathcal{E}_0^\vee \rightarrow \mathcal{O}(-1)$ . Then  $\mathcal{F}$  fits in an exact sequence

$$0 \rightarrow T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}^{\oplus s+2} \rightarrow \mathcal{F} \rightarrow 0,$$

and  $\mathcal{F}$  is a nef vector bundle of rank  $s - 1$  with  $c_1(\mathcal{F}) = 2$ . Moreover, as we have seen in [Ohn14, Remark 6.7],  $\mathcal{F} \cong \Omega_{\mathbb{P}^3}(2)$  if  $s = 4$  and  $\mathcal{F} \cong \mathcal{N}(1)$  if  $s = 3$  where  $\mathcal{N}$  is a null correlation bundle on  $\mathbb{P}^3$ . Hence  $\mathcal{E}$  is either  $\mathcal{O}(1) \oplus \Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}^{\oplus r-4}$  or  $\mathcal{O}(1) \oplus \mathcal{N}(1) \oplus \mathcal{O}^{\oplus r-3}$  if  $Z = \emptyset$ .

**7.2.3. The case where  $c_3 = 4$ .** Suppose that  $c_3 = 4$ . Then  $h^1(\mathcal{E}(-2)) = 0$  and  $h^0(\mathcal{E}(-1)) = 2$  by (7.3) and (7.5) respectively. Hence  $h^q(\mathcal{E}(-2)) = 0$  for all  $q$  by (7.2) and (3.1). It follows from (7.1) that  $h^q(\mathcal{E}|_{L^2}(-2)) = 0$  unless  $q = 1$  and that  $h^1(\mathcal{E}|_{L^2}(-2)) = 1$ . Hence  $h^q(\mathcal{E}(-3)) = 0$  unless  $q = 2$ , and  $h^2(\mathcal{E}(-3)) = 1$ . We shall apply to  $\mathcal{E}$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

It follows from (7.4), (3.9), and (3.2) that  $\text{Ext}^q(G, \mathcal{E}) = 0$  for  $q = 1, 3$ , that  $\text{Ext}^2(G, \mathcal{E}) \cong S_3$ , and that  $\text{Hom}(G, \mathcal{E})$  has, as in (2.3), a projective resolution of the form

$$0 \rightarrow P_0^{\oplus 8} \rightarrow P_1^{\oplus 2} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}) \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E})$ . Hence  $E_2^{-3,2} \cong \mathcal{O}(-1)$  and  $E_2^{p,q} = 0$  unless  $(p, q) = (-3, 2)$ ,  $(-1, 0)$ , or  $(0, 0)$ ; thus  $E_3^{-3,2} \cong \mathcal{O}(-1)$  and  $E_3^{0,0} \cong E_2^{0,0}$ . The Bondal spectral sequence above then induces an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow E_2^{0,0} \rightarrow \mathcal{E} \rightarrow 0$$

and shows that  $E_2^{-1,0} = 0$ , which implies that there exists an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 8} \rightarrow \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0.$$

Since  $\mathcal{O}(-1)$  and  $\mathcal{E}$  are vector bundles, so is  $E_2^{0,0}$ . Hence  $E_2^{0,0} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus e_{0,0}-8}$  by Lemma 4.3; thus  $e_{0,0} - 8 = r - 1$  and we obtain the case (6) of Theorem 1.1.

**7.3. The case where  $n \geq 4$ .** Suppose that  $n \geq 4$ . In this case,  $c_3 = c_3(\mathcal{E}) = 2$  or  $4$ , since  $c_3(\mathcal{E}) = c_3(\mathcal{E}|_{L^3}) = 2$  or  $4$  for any 3-dimensional linear subspace  $L^3$  in  $\mathbb{P}^n$  as is shown in §7.2.

7.3.1. *The case where  $c_3 = 2$ .* Suppose that  $c_3 = 2$ . We shall apply to  $\mathcal{E}(1)$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q = 0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

We first deal with the case  $n = 4$ . For a hyperplane  $H$  in  $\mathbb{P}^4$ ,  $\mathcal{E}|_H$  is in the case (7) of Theorem 1.1, as we have seen in § 7.2.2. Hence

$$H^q(\mathcal{E}|_H(-2)) = \begin{cases} K & \text{if } q = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H^q(\mathcal{E}|_H(-1)) = \begin{cases} K & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $H^q(\mathcal{E}(-k)) = 0$  if  $q > 0$  and  $k \leq 1$  by (3.2). By substituting  $c_2 = 4$  and  $c_3 = 2$  to (3.24), we have

$$0 \leq h^0(\mathcal{E}(-1)) = \frac{1 - c_4}{6}.$$

Hence  $c_4 \leq 1$  and  $c_4 \equiv 1 \pmod{6}$ . Together with (3.7), this implies that  $c_4 = 1$  and that  $h^0(\mathcal{E}(-1)) = 0$ . Hence  $h^q(\mathcal{E}(-2)) = 0$  unless  $q = 1$  and  $h^1(\mathcal{E}(-2)) = 1$ . Thus  $h^q(\mathcal{E}(-3)) = 0$  unless  $q = 1$  or  $2$ . Therefore  $\text{Ext}^q(G, \mathcal{E}(1)) = 0$  for all  $q \geq 3$ , the right  $A$ -module  $\text{Hom}(G, \mathcal{E}(1))$  has, as in (2.3), a projective resolution of the form

$$0 \rightarrow P_0^{\oplus 5e_{0,1}} \rightarrow P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}(1)) \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(1))$  and  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(1))$ , and we have two cases:

- (i)  $(h^1(\mathcal{E}(-3)), h^2(\mathcal{E}(-3))) = (1, 1)$ ;
- (ii)  $(h^1(\mathcal{E}(-3)), h^2(\mathcal{E}(-3))) = (0, 0)$ .

(i) Suppose that  $(h^1(\mathcal{E}(-3)), h^2(\mathcal{E}(-3))) = (1, 1)$ . Then  $\text{Ext}^2(G, \mathcal{E}(1)) \cong S_4$ , and  $\text{Ext}^1(G, \mathcal{E}(1))$  fits in an exact sequence

$$0 \rightarrow S_3 \rightarrow \text{Ext}^1(G, \mathcal{E}(1)) \rightarrow S_4 \rightarrow 0$$

of right  $A$ -modules. Thus it follows from (2.4) that  $\text{Ext}^2(G, \mathcal{E}(1)) \otimes_A^L G \cong \mathcal{O}(-1)[4]$  and that there exists a distinguished triangle

$$\mathcal{O}(-1)[3] \rightarrow T_{\mathbb{P}^4}(-2)[3] \rightarrow \text{Ext}^1(G, \mathcal{E}(1)) \otimes_A^L G \rightarrow .$$

Hence we obtain an isomorphism  $E_2^{-4,2} \cong \mathcal{O}(-1)$ , exact sequences

$$\begin{aligned} 0 \rightarrow E_2^{-4,1} \rightarrow \mathcal{O}(-1) \rightarrow T_{\mathbb{P}^4}(-2) \rightarrow E_2^{-3,1} \rightarrow 0, \\ 0 \rightarrow E_2^{-1,0} \rightarrow \mathcal{O}^{\oplus 5e_{0,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0, \end{aligned}$$

and other vanishing  $E_2$ -terms. Then the Bondal spectral sequence implies that  $E_2^{-4,1} = 0$ , that  $E_3^{-3,1} = 0$ , that  $E_3^{-1,0} \cong \mathcal{O}(-1)$ , and that  $E_2^{0,0} \cong \mathcal{E}(1)$ . Therefore we have the following exact sequences:

$$\begin{aligned} 0 \rightarrow E_2^{-3,1} \rightarrow E_2^{-1,0} \rightarrow \mathcal{O}(-1) \rightarrow 0; \\ 0 \rightarrow E_2^{-1,0} \rightarrow \mathcal{O}^{\oplus 5e_{0,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E}(1) \rightarrow 0. \end{aligned}$$

Since  $\mathcal{E}(1)$  is a vector bundle, so is  $E_2^{-1,0}$ , so that so is  $E_2^{-3,1}$ . On the other hand,  $E_2^{-3,1}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow T_{\mathbb{P}^4}(-2) \rightarrow E_2^{-3,1} \rightarrow 0,$$

which implies that  $E_2^{-3,1}$  cannot be a vector bundle by Lemma 4.1. This is a contradiction. Hence the case where  $(h^1(\mathcal{E}(-3)), h^2(\mathcal{E}(-3))) = (1, 1)$  does not happen.

(ii) Suppose that  $(h^1(\mathcal{E}(-3)), h^2(\mathcal{E}(-3))) = (0, 0)$ . Then we see that  $\text{Ext}^2(G, \mathcal{E}(1)) = 0$  and that  $\text{Ext}^1(G, \mathcal{E}(1)) \cong S_3$ . Since  $S_3 \otimes_A^\mathbb{L} G \cong T_{\mathbb{P}^4}(-2)[3]$  by (2.4), we infer that  $E_2^{p,1} = 0$  unless  $p = -3$  and that  $E_2^{-3,1} \cong T_{\mathbb{P}^4}(-2)$ . Therefore the Bondal spectral sequence induces the following exact sequence

$$0 \rightarrow T_{\mathbb{P}^4}(-2) \rightarrow \mathcal{O}^{\oplus 5e_{0,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since  $\mathcal{E}(1)$  does not admit  $\mathcal{O}$  as a quotient, it follows from [Ohn14, Proposition 2.7] that the sequence above can be reduced to the following form

$$0 \rightarrow T_{\mathbb{P}^4}(-3) \rightarrow \mathcal{O}(-1)^{\oplus 5e_{0,1} - e_{0,0}} \rightarrow \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E} \rightarrow 0.$$

Hence  $5e_{0,1} - e_{0,0} = 10$  and  $e_{0,1} = r + 6$ ; thus the sequence above extends to the case (8) of Theorem 1.1.

**Remark 7.2.** Suppose that  $\mathcal{E}$  is in the case (8) of Theorem 1.1. Then  $\mathcal{E}$  has  $\mathcal{O}^{\oplus r-4}$  as a subbundle; let  $\mathcal{E}_0$  be the quotient bundle  $\mathcal{E}/\mathcal{O}^{\oplus r-4}$  of rank four. In [AM13, §6 III (a)], it is stated that  $\mathcal{E}_0 \cong \Omega_{\mathbb{P}^4}(2)$ . (Therefore we see that  $\mathcal{E} \cong \Omega_{\mathbb{P}^4}(2) \oplus \mathcal{O}^{\oplus r-4}$ .)

For the sake of completeness, we give a different proof of this result in our context. First note that  $\mathcal{E}_0$  fits in an exact sequence

$$0 \rightarrow T_{\mathbb{P}^4}(-3) \rightarrow \mathcal{O}(-1)^{\oplus 10} \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \mathcal{E}_0 \rightarrow 0.$$

Therefore we obtain the following exact sequence

$$0 \rightarrow \mathcal{E}_0^\vee(-1) \rightarrow \mathcal{O}(-1)^{\oplus 10} \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \Omega_{\mathbb{P}^4}(2) \rightarrow 0.$$

We split this sequence into the following two exact sequences:

$$(7.12) \quad 0 \rightarrow \mathcal{E}_0^\vee(-1) \rightarrow \mathcal{O}(-1)^{\oplus 10} \rightarrow \mathcal{G} \rightarrow 0;$$

$$(7.13) \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \Omega_{\mathbb{P}^4}(2) \rightarrow 0.$$

We claim here that the induced map  $H^0(\mathcal{O}^{\oplus 10}) \rightarrow H^0(\Omega_{\mathbb{P}^4}(2))$  is an isomorphism. Since  $h^0(\Omega_{\mathbb{P}^4}(2)) = 10$  by the Bott formula [OSS80, p. 8], it is enough to show that the map is injective. Suppose, to the contrary, that there exists a subbundle  $\mathcal{O} \rightarrow \mathcal{O}^{\oplus 10}$  such that the composite  $\mathcal{O} \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \Omega_{\mathbb{P}^4}(2)$  is zero. Then the subbundle morphism  $\mathcal{O} \rightarrow \mathcal{O}^{\oplus 10}$  induces a subbundle morphism  $\mathcal{O} \rightarrow \mathcal{G}$ ; let  $\mathcal{G}_0$  be the quotient bundle  $\mathcal{G}/\mathcal{O}$ . The composite of the subbundle morphism  $\mathcal{O} \rightarrow \mathcal{G}$  and the extension class in  $\text{Ext}^1(\mathcal{G}, \mathcal{E}_0^\vee(-1))$  corresponding to (7.12) lies in  $H^1(\mathcal{E}_0^\vee(-1))$ , and it gives rise to an exact sequence

$$(7.14) \quad 0 \rightarrow \mathcal{E}_0^\vee \rightarrow \mathcal{F} \rightarrow \mathcal{O}(1) \rightarrow 0.$$

Then  $\mathcal{F}$  is a vector bundle, and it also fits in an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \mathcal{G}_0(1) \rightarrow 0.$$

Hence  $\mathcal{F}^\vee$  is nef. On the other hand, it follows from (7.14) that  $c_3(\mathcal{F}^\vee) = -2$  since  $c_2(\mathcal{E}) = 4$  and  $c_3(\mathcal{E}) = 2$ . This contradicts the non-negativity of the Chern classes of nef vector bundles. Therefore the claim holds. Hence we may assume that the dual of (7.13) is nothing but the exact sequence induced by the two wedge  $\wedge^2(\mathcal{O}^{\oplus 5}) \rightarrow \wedge^2(T_{\mathbb{P}^4}(-1))$  of the surjection in the Euler exact sequence

$$(7.15) \quad 0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus 5} \rightarrow T_{\mathbb{P}^4}(-1) \rightarrow 0.$$

In particular,  $\mathcal{G} \cong \Omega_{\mathbb{P}^4}^2(2)$ . Thus the exact sequence (7.12) implies an exact sequence

$$(7.16) \quad 0 \rightarrow \mathcal{E}_0^\vee \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \Omega_{\mathbb{P}^4}^2(3) \rightarrow 0.$$

Next we claim that the induced map  $H^0(\mathcal{O}^{\oplus 10}) \rightarrow H^0(\Omega_{\mathbb{P}^4}^2(3))$  is an isomorphism. Since  $h^0(\Omega_{\mathbb{P}^4}^2(3)) = 10$  by the Bott formula, it is enough to show that the map is injective. Suppose, to the contrary, that there exists a subbundle  $\mathcal{O} \rightarrow \mathcal{O}^{\oplus 10}$  such that the composite  $\mathcal{O} \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \Omega_{\mathbb{P}^4}^2(3)$  is zero. Then the subbundle morphism  $\mathcal{O} \rightarrow \mathcal{O}^{\oplus 10}$  induces a subbundle morphism  $\mathcal{O} \rightarrow \mathcal{E}_0^\vee$ ; let  $\mathcal{E}_1^\vee$  be the quotient bundle  $\mathcal{E}_0^\vee/\mathcal{O}$ . Then  $\mathcal{E}_1^\vee$  fits in an exact sequence

$$0 \rightarrow \mathcal{E}_1^\vee \rightarrow \mathcal{O}^{\oplus 9} \rightarrow \Omega_{\mathbb{P}^4}^2(3) \rightarrow 0.$$

Hence it follows from the Bott formula that  $h^0(\mathcal{E}_1) = 9$ . Since  $h^0(\mathcal{E}_0) = 10$ , this implies that  $\mathcal{E}_0 \cong \mathcal{E}_1 \oplus \mathcal{O}$ . Thus  $c_4(\mathcal{E}_0) = 0$ , which however contradicts our previous observation that  $c_4 = 1$ . Therefore  $H^0(\mathcal{O}^{\oplus 10}) \rightarrow H^0(\Omega_{\mathbb{P}^4}^2(3))$  is an isomorphism, and we conclude that the exact sequence (7.16) is nothing but the exact sequence induced by the two wedge  $\wedge^2(\mathcal{O}^{\oplus 5}) \rightarrow \wedge^2(T_{\mathbb{P}^4}(-1))$  of the surjection in the Euler exact sequence (7.15). Therefore  $\mathcal{E}_0 \cong \Omega_{\mathbb{P}^4}(2)$ .

Next we deal with the case  $n = 5$ . If  $n = 5$ , then  $h^q(\mathcal{E}(-k)) = 0$  if  $q > 0$  and  $k \leq 2$  by (3.2). Moreover, for any hyperplane  $H$  in  $\mathbb{P}^5$ ,  $h^q(\mathcal{E}|_H(-1)) = 0 = h^q(\mathcal{E}|_H(-3))$  for all  $q$ ,  $h^q(\mathcal{E}|_H(-2)) = 0$  unless  $q = 1$ , and  $h^1(\mathcal{E}|_H(-2)) = 1$  by the consideration in case  $n = 4$ . Since  $h^q(\mathcal{E}(-2)) = 0$  for all  $q$ , this implies that  $h^q(\mathcal{E}(-1)) = 0$  for all  $q$ , that  $h^q(\mathcal{E}(-3)) = 0$  unless  $q = 2$ , and that  $h^2(\mathcal{E}(-3)) = 1$ ; consequently  $h^q(\mathcal{E}(-4)) = 0$  unless  $q = 2$  and  $h^2(\mathcal{E}(-4)) = 1$ . Summing up, we have  $\text{Ext}^q(G, \mathcal{E}(1)) = 0$  unless  $q = 2$  or  $0$ , and  $\text{Ext}^2(G, \mathcal{E}(1))$  fits in an exact sequence

$$0 \rightarrow S_4 \rightarrow \text{Ext}^2(G, \mathcal{E}(1)) \rightarrow S_5 \rightarrow 0$$

of right  $A$ -modules. Moreover  $\text{Hom}(G, \mathcal{E}(1))$  has, as in (2.3), a projective resolution

$$0 \rightarrow P_0^{\oplus 6e_{0,1}} \rightarrow P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}(1)) \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(1))$  and  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(1))$ . Since  $S_4 \otimes_A^L G$  is isomorphic to  $T_{\mathbb{P}^5}(-2)[4]$  and  $S_5 \otimes_A^L G$  to  $\mathcal{O}(-1)[5]$  by (2.4),  $\text{Ext}^2(G, \mathcal{E}(1)) \otimes_A^L G$  fits in a distinguished triangle

$$\mathcal{O}(-1)[4] \rightarrow T_{\mathbb{P}^5}(-2)[4] \rightarrow \text{Ext}^2(G, \mathcal{E}(1)) \otimes_A^L G \rightarrow .$$

Hence we obtain an exact sequence

$$0 \rightarrow E_2^{-5,2} \rightarrow \mathcal{O}(-1) \rightarrow T_{\mathbb{P}^5}(-2) \rightarrow E_2^{-4,2} \rightarrow 0.$$

We also obtain an exact sequence

$$0 \rightarrow E_2^{-1,0} \rightarrow \mathcal{O}^{\oplus 6e_{0,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0.$$

Note that  $E_2^{p,q}$  vanishes unless  $(p, q) = (-5, 2), (-4, 2), (-1, 0)$  or  $(0, 0)$ . The Bondal spectral sequence then implies that  $E_2^{-5,2} = 0$ , that  $E_2^{-4,2} \cong E_2^{-1,0}$ , and that  $E_2^{0,0} \cong \mathcal{E}(1)$ . Since  $\mathcal{E}$  is a vector bundle, so are  $E_2^{0,0}$ ,  $E_2^{-1,0}$ , and  $E_2^{-4,2}$ . On the other hand, since  $E_2^{-4,2}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow T_{\mathbb{P}^5}(-2) \rightarrow E_2^{-4,2} \rightarrow 0,$$

$E_2^{-4,2}$  cannot be a vector bundle by Lemma 4.1. This is a contradiction. Therefore the case  $n = 5$  cannot happen.

Finally, since the case  $n = 5$  cannot happen, we conclude that the case  $n \geq 5$  cannot happen either.

7.3.2. *The case where  $c_3 = 4$ .* Suppose that  $c_3 = 4$ . In this case, we claim first that

- (a)  $h^0(\mathcal{E}(-1)) = 2$  and  $h^q(\mathcal{E}(-1)) = 0$  for all  $q > 0$ ;
- (b)  $h^q(\mathcal{E}(-k)) = 0$  for all  $q$  if  $1 < k < n$ ;
- (c)  $h^{n-1}(\mathcal{E}(-n)) = 1$  and  $h^q(\mathcal{E}(-n)) = 0$  if  $q \neq n-1$ .

We prove these claims by induction on  $n \geq 3$ . If  $n = 3$ , then, as we have seen in § 7.2.3,  $\mathcal{E}$  lies in the case (6) of Theorem 1.1; we see that the claims hold. Suppose that  $n \geq 4$  and that the claims hold for  $\mathcal{E}|_H$ , where  $H$  is a hyperplane in  $\mathbb{P}^n$ .

We first deal with the claim (a). Since  $H^q(\mathcal{E}(-1)) = 0$  for all  $q > 0$  by (3.2), the induction hypothesis implies that  $H^q(\mathcal{E}(-2)) = 0$  for all  $q \geq 2$ . If  $n \geq 5$ , then  $H^1(\mathcal{E}(-2))$  also vanishes by (3.2), and consequently  $h^0(\mathcal{E}(-1)) = 2$  by the induction hypothesis and (3.1). Suppose that  $n = 4$ . By substituting  $c_2 = 4$  and  $c_3 = 4$  to (3.24), we have

$$0 \leq h^0(\mathcal{E}(-1)) = \frac{16 - c_4}{6}.$$

Therefore  $c_4 \leq 16$  and  $c_4 \equiv 16 \pmod{6}$ . Moreover, by substituting  $c_2 = 4$  and  $c_3 = 4$  to (3.25), we obtain  $c_4 \leq 13$ . These imply  $c_4 \leq 10$ , i.e.,  $H(\mathcal{E})^{r+3} \geq 3 > 0$ ; it follows from (3.3) that  $H^1(\mathcal{E}(-2)) = 0$ . Hence the induction hypothesis together with (3.1) implies that  $h^0(\mathcal{E}(-1)) = 2$ , and consequently  $c_4 = 4$ . Therefore the claim (a) holds.

Next we deal with the claim (b). The induction hypothesis is that  $h^q(\mathcal{E}|_H(-k)) = 0$  for all  $q$  if  $1 < k < n-1$ . Note here that we have already proved that  $h^q(\mathcal{E}(-2)) = 0$  for all  $q$  if  $n \geq 3$ . Therefore we see inductively on  $k$  from  $k = 2$  to  $k = n-1$  that  $h^q(\mathcal{E}(-k)) = 0$  for all  $q$ . Hence the claim (b) holds if  $n \geq 3$ .

Finally we deal with the claim (c). By the claim (b), we see in particular that  $h^q(\mathcal{E}(1-n)) = 0$  for all  $q$  if  $n \geq 3$ . Hence  $H^q(\mathcal{E}(-n)) \cong H^{q-1}(\mathcal{E}|_H(1-n))$  for all  $q$ . Therefore the induction hypothesis implies that the claim (c) also holds for  $n \geq 4$ .

We shall apply to  $\mathcal{E}$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

By the claims above,  $\text{Ext}^q(G, \mathcal{E}) = 0$  unless  $q = 0$  or  $n-1$ ,  $\text{Ext}^{n-1}(G, \mathcal{E}) \cong S_n$ , and  $\text{Hom}(G, \mathcal{E})$  has, as in (2.3), a projective resolution of the form

$$0 \rightarrow P_0^{\oplus 2(n+1)} \rightarrow P_1^{\oplus 2} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}) \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E})$ . Hence  $E_2^{p,q} = 0$  unless  $q = 0$  or  $n-1$ , and it follows from (2.5) that  $E_2^{p,n-1} = 0$  unless  $p = -n$  and that  $E_2^{-n,n-1} \cong \mathcal{O}(-1)$ . Moreover  $E_2^{0,0}$  and  $E_2^{-1,0}$  fit in an exact sequence

$$0 \rightarrow E_2^{-1,0} \rightarrow \mathcal{O}^{\oplus 2(n+1)} \rightarrow \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0.$$

The Bondal spectral sequence then implies that  $E_2^{-1,0} = 0$ , that  $E_n^{-n,n-1} \cong \mathcal{O}(-1)$ , that  $E_2^{0,0} \cong E_n^{0,0}$ , and that  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow E_n^{0,0} \rightarrow \mathcal{E} \rightarrow 0.$$

Hence we obtain the following exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 2(n+1)} \rightarrow \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_n^{0,0} \rightarrow 0.$$

Since  $\mathcal{O}(-1)$  and  $\mathcal{E}$  are vector bundles, so is  $E_n^{0,0}$ . Thus  $E_n^{0,0}$  is isomorphic to  $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus e_{0,0}-2(n+1)}$  by Lemma 4.3, and thus  $e_{0,0} - 2(n+1) = r - 1$ ; we get the case (6) of Theorem 1.1.

## 8. THE CASE WHERE $c_2(\mathcal{E}) = 5$

In this section, we give a proof of Theorem 1.1 in case  $c_2(\mathcal{E}) = 5$ ; throughout this section, we assume that  $c_2(\mathcal{E}) = 5$ .

**8.1. The case where  $n = 2$ .** Suppose that  $n = 2$ . Then  $h^1(\mathcal{E}(-2)) = 2$  by (3.10). Now we apply the Bondal spectral sequence (3.14); it follows from (3.15), (3.16), and Lemma 5.1 (1) that  $H^1(\mathcal{E}(-1)) = 0$ , that  $E_2^{-2,1} \cong \mathcal{O}(-1)^{\oplus 2}$ , and that  $E_2^{-1,1} = 0$ . Thus  $E_3^{0,0} \cong \mathcal{E}$  by (3.16). Hence  $\mathcal{E}$  fits in the following exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow E_2^{0,0} \rightarrow \mathcal{E} \rightarrow 0.$$

Since  $\mathcal{E}$  and  $\mathcal{O}(-1)$  are vector bundles, so is  $E_2^{0,0}$ . The Riemann-Roch formulas (3.11) and (3.12) respectively show that  $h^0(\mathcal{E}(-1)) = 1$  and  $h^0(\mathcal{E}) = 4 + r$ . It then follows from (3.17) that  $E_2^{0,0}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 3} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus 4+r} \rightarrow E_2^{0,0} \rightarrow 0.$$

Since  $E_2^{0,0}$  is a vector bundle, Lemma 4.3 shows that  $E_2^{0,0} \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+1}$ . Therefore we get the case (9) of Theorem 1.1.

**8.2. The case where  $n = 3$ .** Suppose that  $n = 3$ . As we have seen in §8.1, for any plane  $L^2 \subset \mathbb{P}^3$ ,  $\mathcal{E}|_{L^2}$  fits in an exact sequence

$$(8.1) \quad 0 \rightarrow \mathcal{O}_{L^2}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{L^2}(1) \oplus \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E}|_{L^2} \rightarrow 0$$

since  $c_2(\mathcal{E}|_{L^2}) = c_2(\mathcal{E}) = 5$ . We see that  $H^q(\mathcal{E}|_{L^2}(-k)) = 0$  for all  $q > 0$  if  $k \leq 1$ . Since  $H^q(\mathcal{E}(-k)) = 0$  for all  $q > 0$  if  $k \leq 0$  by (3.2), this implies that

$$(8.2) \quad H^q(\mathcal{E}(-k)) = 0 \text{ for all } q \geq 2 \text{ if } k \leq 2.$$

Then it follows from (3.1) and (3.19) that

$$(8.3) \quad 0 \geq -h^1(\mathcal{E}(-2)) = \frac{c_3 - 7}{2}.$$

Hence  $c_3$  is odd and  $c_3 \leq 7$ . On the other hand, we have  $c_3 \geq 3$  by (3.23). Thus  $(h^1(\mathcal{E}(-2)), c_3) = (0, 7)$ ,  $(1, 5)$ , or  $(2, 3)$ . We also have

$$\chi(\mathcal{E}(-1)) = \frac{c_3 - 5}{2}$$

by (3.20). Note here that  $H(\mathcal{E})$  is nef and big if  $c_3 = 5$  or  $7$ . Hence we see that

$$(8.4) \quad H^1(\mathcal{E}(-1)) = 0 \text{ if } c_3 = 5 \text{ or } 7$$

by (3.3). Thus the formula above implies that

$$(8.5) \quad h^0(\mathcal{E}(-1)) = \frac{c_3 - 5}{2}$$

if  $c_3 = 5$  or  $7$ .

8.2.1. *The case where  $c_3 = 3$ .* Suppose that  $c_3 = 3$ . Then  $\chi(\mathcal{E}(-1)) = -1$  and

$$(8.6) \quad h^1(\mathcal{E}(-2)) = 2.$$

We apply to  $\mathcal{E}(1)$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q = 0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

It follows from (8.2) that  $E_2^{p,q} = 0$  for all  $p$  if  $q \geq 2$ . By (3.21) and (3.22), we have

$$(8.7) \quad h^0(\mathcal{E}) = r + 3 \text{ and } h^0(\mathcal{E}(1)) = 4r + 10.$$

Note that there exist an exact sequence

$$0 \rightarrow H^0(\mathcal{E}(-1)) \rightarrow H^0(\mathcal{E}|_{L^2}(-1)) \rightarrow H^1(\mathcal{E}(-2)) \rightarrow H^1(\mathcal{E}(-1)) \rightarrow 0$$

and an isomorphism  $H^0(\mathcal{E}|_{L^2}(-1)) \cong K$ . Therefore we have two cases:

- (i)  $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (0, 1)$ ;
- (ii)  $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (1, 2)$ .

(i) Suppose that  $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (0, 1)$ . Then it follows from the assumption  $h^1(\mathcal{E}(-1)) = 1$  and (8.6) that the  $A$ -module  $\text{Ext}^1(G, \mathcal{E}(1))$  fits in an exact sequence

$$0 \rightarrow S_2 \rightarrow \text{Ext}^1(G, \mathcal{E}(1)) \rightarrow S_3^{\oplus 2} \rightarrow 0.$$

Hence we get, by (2.4), a distinguished triangle

$$\mathcal{O}(-1)^{\oplus 2}[2] \rightarrow T_{\mathbb{P}^3}(-2)[2] \rightarrow \text{Ext}^1(G, \mathcal{E}(1)) \otimes_A^L G \rightarrow .$$

Thus  $E_2^{p,1} = 0$  unless  $p = -3$  or  $-2$ , and we have an exact sequence

$$(8.8) \quad 0 \rightarrow E_2^{-3,1} \rightarrow \mathcal{O}(-1)^{\oplus 2} \xrightarrow{\varphi} T_{\mathbb{P}^3}(-2) \rightarrow E_2^{-2,1} \rightarrow 0.$$

As in (2.3),  $\text{Hom}(G, \mathcal{E}(1))$  fits in an exact sequence

$$0 \rightarrow P_0^{\oplus 4r+12} \rightarrow P_1^{\oplus r+3} \oplus P_0^{\oplus 4r+10} \rightarrow \text{Hom}(G, \mathcal{E}(1)) \rightarrow 0,$$

since we have (8.7) and  $h^0(\mathcal{E}(-1)) = 0$  by the assumption. Thus  $E_2^{p,0} = 0$  if  $p < -1$ , and we obtain an exact sequence

$$(8.9) \quad 0 \rightarrow E_2^{-1,0} \rightarrow \mathcal{O}^{\oplus 4r+12} \rightarrow \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow E_2^{0,0} \rightarrow 0.$$

The Bondal spectral sequence then implies that  $E_2^{-3,1} \cong E_2^{-1,0}$  and that  $\mathcal{E}(1)$  fits in an exact sequence

$$(8.10) \quad 0 \rightarrow E_2^{-2,1} \rightarrow E_2^{0,0} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Let  $k$  be the rank of the composite  $\mathcal{O}^{\oplus 4r+12} \rightarrow \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{O}^{\oplus 4r+10}$  of the morphism  $\mathcal{O}^{\oplus 4r+12} \rightarrow \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10}$  in (8.9) and the projection  $\mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{O}^{\oplus 4r+10}$ . Then the exact sequence (8.9) together with  $E_2^{-3,1} \cong E_2^{-1,0}$  implies the following exact sequence

$$(8.11) \quad 0 \rightarrow E_2^{-3,1} \rightarrow \mathcal{O}^{\oplus 4r+12-k} \rightarrow \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10-k} \rightarrow E_2^{0,0} \rightarrow 0.$$

Since the composite  $\mathcal{O}^{\oplus 4r+12-k} \rightarrow \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10-k} \rightarrow \mathcal{O}^{\oplus 4r+10-k}$  is zero, we have a surjection  $E_2^{0,0} \rightarrow \mathcal{O}^{\oplus 4r+10-k}$ . Denote by  $\mathcal{K}$  the kernel of this surjection. Then we have an exact sequence

$$(8.12) \quad 0 \rightarrow \mathcal{K} \rightarrow E_2^{0,0} \rightarrow \mathcal{O}^{\oplus 4r+10-k} \rightarrow 0.$$

We split the sequence (8.11) into the following two exact sequences:

$$(8.13) \quad 0 \rightarrow E_2^{-3,1} \rightarrow \mathcal{O}^{\oplus 4r+12-k} \rightarrow \mathcal{G} \rightarrow 0;$$

$$(8.14) \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10-k} \rightarrow E_2^{0,0} \rightarrow 0.$$

By the snake lemma,  $\mathcal{K}$  fits in an exact sequence

$$(8.15) \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}(1)^{\oplus r+3} \rightarrow \mathcal{K} \rightarrow 0.$$

Thus  $H^1(\mathcal{K}) \cong H^2(\mathcal{G}) \cong H^3(E_2^{-3,1})$ . This implies that if  $H^3(E_2^{-3,1}) = 0$  then  $E_2^{0,0} \cong \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+10-k}$  by (8.12). We shall divide the case into three cases:  $\dim \operatorname{Im} H^0(\varphi(1)) = 0$ , 1, or 2, where  $\varphi$  is the morphism in (8.8).

Suppose that  $\dim \operatorname{Im} H^0(\varphi(1)) = 0$ , i.e., that  $\varphi = 0$ . Then  $E_2^{-3,1} \cong \mathcal{O}(-1)^{\oplus 2}$  and  $E_2^{0,0} \cong T_{\mathbb{P}^3}(-2)$ . Hence  $E_2^{0,0}$  fits in an exact sequence

$$0 \rightarrow T_{\mathbb{P}^3}(-2) \rightarrow E_2^{0,0} \rightarrow \mathcal{E}(1) \rightarrow 0,$$

and thus  $E_2^{0,0}$  is a vector bundle. Since  $E_2^{0,0}$  is a vector bundle, so is  $\mathcal{G}$  by (8.14). Then the morphism  $E_2^{-3,1} \rightarrow \mathcal{O}^{\oplus 4r+12-k}$  in (8.13) is a subbundle morphism. In particular, the composite  $\mathcal{O}(-1) \rightarrow \mathcal{O}(-1)^{\oplus 2} \cong E_2^{-3,1} \rightarrow \mathcal{O}^{\oplus 4r+12-k}$  is a subbundle morphism. Therefore  $4r + 12 - k \geq 4$ . Thus  $4r + 10 - k \geq 2$ . Since  $H^3(E_2^{-3,1}) = H^3(\mathcal{O}(-1)^{\oplus 2}) = 0$ ,  $E_2^{0,0} \cong \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+10-k}$ . Therefore we have an exact sequence

$$0 \rightarrow T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+10-k} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since  $\mathcal{E}(1)$  cannot admit  $\mathcal{O}$  as a quotient, any composite

$$T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+10-k} \rightarrow \mathcal{O}$$

is non-zero. If the cokernel of the composite  $T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}$  is non-zero, then it is isomorphic to  $\mathcal{O}_L$  for a line  $L$  in  $\mathbb{P}^3$  by Lemma 4.2. This implies that  $\mathcal{E}|_L(1)$  admits  $\mathcal{O}_L$  as a quotient, which contradicts that  $\mathcal{E}$  is nef. Therefore the cokernel of the composite  $T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}$  is zero, i.e., the composite  $T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}$  is surjective. Note here that the kernel of the surjection  $T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}$  is isomorphic to  $\mathcal{N}(-1)$ , where  $\mathcal{N}$  is a null correlation bundle. Therefore we get an exact sequence

$$0 \rightarrow \mathcal{N}(-1) \rightarrow \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+9-k} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since  $\mathcal{E}(1)$  cannot admit  $\mathcal{O}$  as a quotient, any composite  $\mathcal{N}(-1) \rightarrow \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+9-k} \rightarrow \mathcal{O}$  cannot be zero. Note here that  $\mathcal{N} \cong \mathcal{N}^\vee$  and that the zero locus  $Z$  of any non-zero global section  $s$  of  $\mathcal{N}(1)$  is non-empty since  $c_2(\mathcal{N}(1)) = 2$ . This implies that the zero locus  $Z$  of  $s$  has dimension  $\geq 1$ . Hence  $\mathcal{E}|_Z(1)$  admits  $\mathcal{O}_Z$  as a quotient, which contradicts that  $\mathcal{E}$  is nef. Therefore the case where  $\varphi = 0$  does not happen.

Suppose that  $\dim \operatorname{Im} H^0(\varphi(1)) = 1$ . Then we have an isomorphism  $E_2^{-3,1} \cong \mathcal{O}(-1)$  and an exact sequence

$$(8.16) \quad 0 \rightarrow \mathcal{O}(-1) \rightarrow T_{\mathbb{P}^3}(-2) \rightarrow E_2^{-2,1} \rightarrow 0.$$

Note here that  $E_2^{-2,1}$  is locally free except for one point  $p$  by Lemma 4.1, and consequently so is  $E_2^{0,0}$  by (8.10); thus so is  $\mathcal{G}$  by (8.14). Hence the morphism  $\mathcal{O}(-1) = E_2^{-3,1} \rightarrow \mathcal{O}^{\oplus 4r+12-k}$  in (8.13) is a subbundle morphism except for the point  $p$ . Thus  $4r + 12 - k \geq 3$ , and consequently  $4r + 10 - k \geq 1$ . Since  $H^3(E_2^{-3,1}) = H^3(\mathcal{O}(-1)) = 0$ , we have an



isomorphism  $E_2^{0,0} \cong \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+10-k}$ . Note here that  $\mathcal{K}$  is locally free except for the point  $p$ . By combining (8.10) with (8.16), we obtain an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+10-k} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

As in the case  $\varphi = 0$ , the assumption that  $\mathcal{E}$  is nef imposes that any composite  $T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+10-k} \rightarrow \mathcal{O}$  must be surjective. Hence we get an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{N}(-1) \rightarrow \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+9-k} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

On the other hand, any map  $\mathcal{O}(-1) \rightarrow \mathcal{N}(-1)$  cannot be injective since  $H^0(\mathcal{N}) = 0$ . This is a contradiction. Therefore the case  $\dim \operatorname{Im} H^0(\varphi(1)) = 1$  does not occur.

Suppose that  $\dim \operatorname{Im} H^0(\varphi(1)) = 2$ : let  $(s, t)$  be a basis of  $\operatorname{Im} H^0(\varphi(1))$ . Since the zero locus of any non-zero global section of  $T_{\mathbb{P}^3}(-1)$  has codimension  $\geq 2$ , the quotient sheaf  $T_{\mathbb{P}^3}(-1)/\mathcal{O}s$  is torsion-free. Hence  $t$  induces an injection from  $\mathcal{O}$  to the quotient  $T_{\mathbb{P}^3}(-1)/\mathcal{O}s$ . This implies that  $\varphi$  is injective. Hence  $E_2^{-3,1} = 0$ , and thus  $\mathcal{G} \cong \mathcal{O}^{\oplus 4r+12-k}$  by (8.13). Then it follows from (8.15) that  $\mathcal{K}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 4r+12-k} \rightarrow \mathcal{O}(1)^{\oplus r+3} \rightarrow \mathcal{K} \rightarrow 0.$$

We have  $H^1(\mathcal{K}) = 0$  and thus  $E_2^{0,0} \cong \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+10-k}$ . Combining (8.10) with (8.8) we obtain an exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow T_{\mathbb{P}^3}(-2) \rightarrow \mathcal{K} \oplus \mathcal{O}^{\oplus 4r+10-k} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

If  $4r + 10 - k \geq 1$ , then we get a contradiction by the (almost) same argument as in the case where  $\dim \operatorname{Im} H^0(\varphi(1)) = 1$ . Hence  $4r + 10 - k = 0$  and  $E_2^{0,0} \cong \mathcal{K}$ . Since we have a surjection  $\mathcal{O}(1)^{\oplus r+3} \rightarrow \mathcal{K}$ , this implies that  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{O}^{\oplus r+3} \rightarrow \mathcal{E} \rightarrow 0$$

for some vector bundle  $\mathcal{H}$  of rank three. Note here that the dual  $\mathcal{H}^\vee$  of  $\mathcal{H}$  is globally generated and thus nef. Moreover  $c_1(\mathcal{H}^\vee) = 3$  and

$$c_2(\mathcal{H}^\vee) = c_2(\mathcal{H}) = -c_1(\mathcal{H})c_1(\mathcal{E}) - c_2(\mathcal{E}) = 4.$$

Furthermore

$$c_3(\mathcal{H}^\vee) = -c_3(\mathcal{H}) = c_2(\mathcal{H})c_1(\mathcal{E}) + c_1(\mathcal{H})c_2(\mathcal{E}) + c_3(\mathcal{E}) = 0.$$

On the other hand, as we have seen in §7.2.1, no nef vector bundle on  $\mathbb{P}^3$  exists in case  $(c_1, c_2, c_3) = (3, 4, 0)$ . This is a contradiction. Therefore the case  $\dim \operatorname{Im} H^0(\varphi(1)) = 2$  does not occur either.

By the consideration above, we conclude that the case  $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (0, 1)$  does not happen.

(ii) Suppose that  $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (1, 2)$ . The assumption  $H^1(\mathcal{E}(-1)) = 2$  together with (8.6) implies that the  $A$ -module  $\operatorname{Ext}^1(G, \mathcal{E}(1))$  fits in an exact sequence

$$0 \rightarrow S_2^{\oplus 2} \rightarrow \operatorname{Ext}^1(G, \mathcal{E}(1)) \rightarrow S_3^{\oplus 2} \rightarrow 0.$$

Hence we get, by (2.4), a distinguished triangle

$$\mathcal{O}(-1)^{\oplus 2}[2] \rightarrow T_{\mathbb{P}^3}(-2)^{\oplus 2}[2] \rightarrow \operatorname{Ext}^1(G, \mathcal{E}(1)) \otimes_A^L G \rightarrow .$$

Thus  $E_2^{p,1} = 0$  unless  $p = -3$  or  $-2$ , and there exists an exact sequence

$$0 \rightarrow E_2^{-3,1} \rightarrow \mathcal{O}(-1)^{\oplus 2} \xrightarrow{\varphi} T_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow E_2^{-2,1} \rightarrow 0.$$

Set  $a = \dim \operatorname{Im} H^0(\varphi(1))$ . Then  $E_2^{-3,1} \cong \mathcal{O}(-1)^{\oplus 2-a}$  as in the case (i), and the sequence above induces an exact sequence

$$(8.17) \quad 0 \rightarrow \mathcal{O}(-1)^{\oplus a} \rightarrow T_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow E_2^{-2,1} \rightarrow 0.$$

It follows from (2.2), (8.7), and the assumption  $h^0(\mathcal{E}(-1)) = 1$  that  $\operatorname{Hom}(G, \mathcal{E}(1))$  fits in an exact sequence

$$0 \rightarrow P_0^{\oplus 16} \rightarrow P_1^{\oplus 4} \oplus P_0^{\oplus 4r+22} \rightarrow P_2 \oplus P_1^{\oplus r+3} \oplus P_0^{\oplus 4r+10} \rightarrow \operatorname{Hom}(G, \mathcal{E}(1)) \rightarrow 0.$$

Hence  $E_2^{p,0} = 0$  if  $p < -2$ . The sequence above induces a complex

$$0 \rightarrow \mathcal{O}^{\oplus 16} \rightarrow \mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}^{\oplus 4r+22} \xrightarrow{\psi} \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow E_2^{0,0} \rightarrow 0.$$

Now the Bondal spectral sequence implies that  $E_2^{-3,1} \cong E_2^{-1,0}$ , that  $E_2^{-2,0} = 0$ , and that there exists an exact sequence

$$0 \rightarrow E_2^{-2,1} \rightarrow E_2^{0,0} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Combining this sequence with (8.17), we obtain an exact sequence

$$(8.18) \quad 0 \rightarrow \mathcal{O}(-1)^{\oplus a} \rightarrow T_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow E_2^{0,0} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since  $E_2^{-1,0} \cong \mathcal{O}(-1)^{\oplus 2-a}$  and  $E_2^{-2,0} = 0$ ,  $\operatorname{Ker} \psi$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 16} \rightarrow \operatorname{Ker} \psi \rightarrow \mathcal{O}(-1)^{\oplus 2-a} \rightarrow 0.$$

Thus  $\operatorname{Ker} \psi \cong \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1)^{\oplus 2-a}$ . Therefore we get an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1)^{\oplus 2-a} &\rightarrow \mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}^{\oplus 4r+22} \\ &\rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow E_2^{0,0} \rightarrow 0. \end{aligned}$$

We split this sequence into the following two exact sequences:

$$(8.19) \quad 0 \rightarrow \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1)^{\oplus 2-a} \rightarrow \mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}^{\oplus 4r+22} \rightarrow \mathcal{G} \rightarrow 0;$$

$$(8.20) \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow E_2^{0,0} \rightarrow 0.$$

Since  $H^q(\Omega_{\mathbb{P}^3}(k)) = 0$  for all  $q > 0$  if  $k = 1, 2$ , or  $3$  by the Bott formula [OSS80, p.8], we obtain  $H^1(\Omega_{\mathbb{P}^3}(2) \otimes \mathcal{G}) = 0$  by (8.19), and thus  $\operatorname{Ext}^1(T_{\mathbb{P}^3}(-2)^{\oplus 2}, \mathcal{G}) = 0$ . Therefore it follows from (8.18) and (8.20) that  $\mathcal{E}(1)$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus a} \rightarrow \mathcal{G} \oplus T_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Combining this sequence with the exact sequence below induced by (8.19)

$$0 \rightarrow \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1)^{\oplus 2-a} \rightarrow \mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}^{\oplus 4r+22} \oplus T_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow \mathcal{G} \oplus T_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow 0,$$

we obtain an exact sequence

$$(8.21) \quad \begin{aligned} 0 \rightarrow \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1)^{\oplus 2} &\rightarrow \mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}^{\oplus 4r+22} \oplus T_{\mathbb{P}^3}(-2)^{\oplus 2} \\ &\rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{E}(1) \rightarrow 0. \end{aligned}$$

We claim here that the composite

$$\begin{aligned} \mathcal{O}(1)^{\oplus 4} &\rightarrow \mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}^{\oplus 4r+22} \oplus T_{\mathbb{P}^3}(-2)^{\oplus 2} \\ &\rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{O}(1)^{\oplus r+3} \end{aligned}$$

is injective. To see this, we first split the sequence (8.21) into the following two exact sequences:

$$\begin{aligned} 0 \rightarrow \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1)^{\oplus 2} \rightarrow \mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}^{\oplus 4r+22} \oplus T_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow \mathcal{H} \rightarrow 0; \\ 0 \rightarrow \mathcal{H} \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{E}(1) \rightarrow 0. \end{aligned}$$

Since the composite  $\mathcal{O}(2) \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{E}(1)$  is injective, it extends to an exact sequence

$$0 \rightarrow \mathcal{O}(2) \rightarrow \mathcal{E}(1) \rightarrow \mathcal{F} \rightarrow 0$$

for some coherent sheaf  $\mathcal{F}$  on  $\mathbb{P}^3$ . Thus we obtain an exact sequence

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{O}(1)^{\oplus r+3} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{F} \rightarrow 0.$$

Then the composite

$$H^0(\mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1)^{\oplus 4r+22} \oplus T_{\mathbb{P}^3}(-3)^{\oplus 2}) \rightarrow H^0(\mathcal{H}(-1)) \rightarrow H^0(\mathcal{O}^{\oplus r+3} \oplus \mathcal{O}(-1)^{\oplus 4r+10})$$

is injective, and the claim follows. By this claim, the sequence (8.21) can be reduced to the following form

$$\begin{aligned} 0 \rightarrow \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1)^{\oplus 2} \rightarrow \mathcal{O}^{\oplus 4r+22} \oplus T_{\mathbb{P}^3}(-2)^{\oplus 2} \\ \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r-1} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{E}(1) \rightarrow 0. \end{aligned}$$

Since the composite  $\mathcal{O}^{\oplus 16} \rightarrow \mathcal{O}^{\oplus 16} \oplus \mathcal{O}(-1)^{\oplus 2} \rightarrow \mathcal{O}^{\oplus 4r+22} \oplus T_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow \mathcal{O}^{\oplus 4r+22}$  is injective, we conclude that the sequence above can be reduced to the following

$$0 \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow \mathcal{O}^{\oplus 4r+6} \oplus T_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r-1} \oplus \mathcal{O}^{\oplus 4r+10} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

This is the case (11) of Theorem 1.1.

8.2.2. *The case where  $c_3 = 5$ .* Suppose that  $c_3 = 5$ . We shall apply to  $\mathcal{E}(1)$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q = 0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

Note first that  $\text{Ext}^q(G, \mathcal{E}(1)) = 0$  for all  $q \geq 2$  by (8.2). Since  $c_3 = 5$ , it follows from (8.3) that  $h^1(\mathcal{E}(-2)) = 1$ . This, together with (8.4) and (3.2), implies that  $\text{Ext}^1(G, \mathcal{E}(1)) \cong S_3$ . Thus  $E_2^{p,q} = 0$  unless  $(p, q) = (-3, 1)$  if  $q > 0$ , and  $E_2^{-3,1} \cong \mathcal{O}(-1)$  by (2.5). Since  $c_3 = 5$ , it follows from (8.5) that  $h^0(\mathcal{E}(-1)) = 0$ . The projective resolution (2.3) with  $d = 1$  and the Bondal spectral sequence then imply that  $\mathcal{E}(1)$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus 4e_{0,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E}(1) \rightarrow 0$$

where  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(1))$  and  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(1))$ . This sequence can be reduced to the exact sequence below by [Ohn14, Proposition 2.7] since  $\mathcal{E}$  is nef

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 4e_{0,1}-e_{0,0}} \rightarrow \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{E} \rightarrow 0.$$

Hence  $4e_{0,1} - e_{0,0} = 5$  and  $e_{0,1} = r + 4$ . This is the case (10) of Theorem 1.1.

8.2.3. *The case where  $c_3 = 7$ .* Suppose that  $c_3 = 7$ . We shall apply to  $\mathcal{E}$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

Since  $c_3 = 7$ , it follows from (8.3) that  $h^1(\mathcal{E}(-2)) = 0$ . Together with (8.2) and (3.1), this implies that  $h^q(\mathcal{E}(-2)) = 0$  for all  $q$ . Since  $h^1(\mathcal{E}|_{L^2}(-2)) = 2$  and  $h^q(\mathcal{E}|_{L^2}(-2)) = 0$  unless  $q = 1$  by (8.1), we infer that  $h^2(\mathcal{E}(-3)) = 2$  and that  $h^q(\mathcal{E}(-3)) = 0$  unless  $q = 2$ . Together with (8.4), (8.2), and (3.2), we therefore conclude that  $\text{Ext}^2(G, \mathcal{E}) \cong S_3^{\oplus 2}$  and that  $\text{Ext}^q(G, \mathcal{E}) = 0$  if  $q = 1$  or  $3$ . Thus  $E_2^{p,q} = 0$  unless  $(p, q) = (-3, 2)$  if  $q > 0$ , and  $E_2^{-3,2} = \mathcal{O}(-1)^{\oplus 2}$  by (2.5). Since  $c_3 = 7$ , it follows from (8.5) that  $h^0(\mathcal{E}(-1)) = 1$ . The projective resolution (2.3) with  $d = 0$  and the Bondal spectral sequence then imply that there exist the following two exact sequences:

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow E_2^{0,0} \rightarrow \mathcal{E} \rightarrow 0; \\ 0 \rightarrow \mathcal{O}^{\oplus 4} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0 \end{aligned}$$

where  $e_{0,0} = \text{hom}(G, \mathcal{E})$ . Hence  $E_2^{0,0}$  is a vector bundle, and Lemma 4.3 then implies that  $E_2^{0,0} \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus e_{0,0}-4}$ ; consequently  $e_{0,0} - 4 = r + 1$ . This is the case (9) of Theorem 1.1.

8.3. **The case where  $n \geq 4$ .** Suppose that  $n \geq 4$ . We divide the case into three cases:  $c_3 = c_3(\mathcal{E}) = 3, 5$ , or  $7$ , since  $c_3(\mathcal{E}) = c_3(\mathcal{E}|_{L^3}) = 3, 5$ , or  $7$  for a 3-dimensional linear subspace  $L^3$  of  $\mathbb{P}^n$ , as we have seen in §8.2.

8.3.1. *The case where  $c_3 = 3$ .* Suppose that  $c_3 = 3$ . We first consider the case  $n = 4$ . By substituting  $c_2 = 5$  and  $c_3 = 3$  to (3.24), we have

$$0 \leq h^0(\mathcal{E}(-1)) = -\frac{15 + c_4}{6}.$$

Hence  $c_4 \leq -15$ . However this contradicts (3.7). Thus this case does not happen. Since the case  $n = 4$  does not happen, neither happens the case  $n \geq 5$ .

8.3.2. *The case where  $c_3 = 5$ .* Suppose that  $c_3 = 5$  and that the restriction  $\mathcal{E}|_H$  of  $\mathcal{E}$  to a hyperplane  $H$  in  $\mathbb{P}^n$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}_H(-2) \rightarrow \mathcal{O}_H(-1)^{\oplus 5} \rightarrow \mathcal{O}_H^{\oplus r+4} \rightarrow \mathcal{E}|_H \rightarrow 0.$$

We shall apply to  $\mathcal{E}(1)$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0 \end{cases}$$

to show that  $\mathcal{E}$  also fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 5} \rightarrow \mathcal{O}^{\oplus r+4} \rightarrow \mathcal{E} \rightarrow 0.$$

First note that  $H^q(\mathcal{E}|_H(-k)) = 0$  for all  $q$  if  $1 \leq k \leq n-3$ . This vanishing and (3.1) imply that  $H^0(\mathcal{E}(-1)) = 0$ . Hence  $H^q(\mathcal{E}(-1)) = 0$  for all  $q$  by (3.2). Therefore we see inductively that  $H^q(\mathcal{E}(-k)) = 0$  for all  $q$  if  $1 \leq k \leq n-2$ . Second note that  $H^q(\mathcal{E}|_H(2-n)) = 0$  unless  $q = n-3$  and that  $h^{n-3}(\mathcal{E}|_H(2-n)) = 1$ . Since  $H^q(\mathcal{E}(2-n)) = 0$  for all  $q$ , this implies that  $H^q(\mathcal{E}(1-n)) = 0$  unless  $q = n-2$  and that  $h^{n-2}(\mathcal{E}(1-n)) = 1$ . Hence it follows from (3.2) that  $\text{Ext}^q(G, \mathcal{E}(1)) = 0$  unless  $q = n-2$  or  $q = 0$  and that  $\text{Ext}^{n-2}(G, \mathcal{E}(1)) \cong S_n$ .

Therefore  $E_2^{p,q} = 0$  unless  $(p, q) = (-n, n-2)$  if  $q > 0$ , and  $E_2^{-n, n-2} \cong \mathcal{O}(-1)$  by (2.5). The Bondal spectral sequence then implies that  $\mathcal{E}(1)$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus(n+1)e_{0,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E}(1) \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(1))$  and  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(1))$ . This sequence can be reduced to the following exact sequence by [Ohn14, Proposition 2.7] since  $\mathcal{E}$  is nef

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus(n+1)e_{0,1}-e_{0,0}} \rightarrow \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{E} \rightarrow 0.$$

Hence  $(n+1)e_{0,1} - e_{0,0} = 5$  and  $e_{0,1} = r+4$ . Note here that the morphism  $\mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 5}$  must be a subbundle morphism. Therefore this case is possible only if  $n = 4$ . This is the case (10) of Theorem 1.1.

**Remark 8.1.** Suppose that  $n = 4$  and that  $\mathcal{E}$  fits in an exact sequence in the case (10) of Theorem 1.1. Then  $\mathcal{E}$  is an extension of the Tango bundle by a trivial bundle  $\mathcal{O}^{\oplus r-3}$ .

The reason is as follows. Since  $\mathcal{E}$  is globally generated,  $\mathcal{E}$  has  $\mathcal{O}^{\oplus r-4}$  as a subbundle; denote by  $\mathcal{E}_0$  the quotient bundle  $\mathcal{E}/\mathcal{O}^{\oplus r-4}$ . Since  $\mathcal{E}_0$  is globally generated of rank four with  $c_4(\mathcal{E}_0) = 0$ ,  $\mathcal{E}_0$  also has  $\mathcal{O}$  as a subbundle; denote by  $\mathcal{E}_1$  the quotient bundle  $\mathcal{E}_0/\mathcal{O}$ . We show that  $\mathcal{E}_1$  is the Tango bundle. First note that the dual  $\mathcal{E}_1^\vee$  of  $\mathcal{E}_1$  fits in an exact sequence

$$0 \rightarrow \mathcal{E}_1^\vee \rightarrow \mathcal{O}^{\oplus 7} \rightarrow \Omega_{\mathbb{P}^4}(2) \rightarrow 0.$$

Note also that  $h^0(\mathcal{E}_1^\vee) = 0$ ; indeed, if  $h^0(\mathcal{E}_1^\vee) \neq 0$ , then  $\mathcal{E}_1$  would admit  $\mathcal{O}$  as a direct summand, which contradicts the fact that  $c_3(\mathcal{E}_1) = 5 \neq 0$  and the rank of  $\mathcal{E}_1$  is three. Since  $h^0(\Omega_{\mathbb{P}^4}(2)) = 10$  by the Bott formula [OSS80, p. 8], this implies that  $h^1(\mathcal{E}_1^\vee) = 3$ . Now we have an isomorphism  $\text{Ext}^1(\text{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O}, \mathcal{E}_1^\vee) \cong \text{End}(\text{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee))$ ; let  $\xi$  be the element in  $\text{Ext}^1(\text{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O}, \mathcal{E}_1^\vee)$  corresponding to the identity in  $\text{End}(\text{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee))$ . Consider the extension

$$0 \rightarrow \mathcal{E}_1^\vee \rightarrow \mathcal{F} \rightarrow \text{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O} \rightarrow 0$$

corresponding to  $\xi$ ; then  $H^0(\mathcal{F}) \cong H^0(\mathcal{E}_1^\vee) = 0$  and  $H^1(\mathcal{F}) = 0$ . Let

$$0 \rightarrow \mathcal{O}^{\oplus 7} \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \text{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O} \rightarrow 0$$

be the extension corresponding to the image of  $\xi$  via the map  $\text{Ext}^1(\text{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O}, \mathcal{E}_1^\vee) \rightarrow \text{Ext}^1(\text{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O}, \mathcal{O}^{\oplus 7})$ . Then  $\mathcal{F}$  fits in an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \Omega_{\mathbb{P}^4}(2) \rightarrow 0.$$

Since  $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$ , the induced map  $H^0(\mathcal{O}^{\oplus 10}) \rightarrow H^0(\Omega_{\mathbb{P}^4}(2))$  is an isomorphism. Therefore  $\mathcal{F}^\vee \cong \Omega_{\mathbb{P}^4}^2(3)$ , and thus  $\mathcal{E}_1$  is the Tango bundle.

According to [KPR03, §4], Trautmann [Tra73] and Vetter [Vet73] give an explicit construction of the bundle which is, up to taking duals and twists by  $\mathcal{O}(1)$ , the Tango bundle.

**8.3.3. The case where  $c_3 = 7$ .** Suppose that  $c_3 = 7$  and that the restriction  $\mathcal{E}|_H$  of  $\mathcal{E}$  to a hyperplane  $H$  in  $\mathbb{P}^n$  lies in the case (9) of Theorem 1.1. Then we shall show that  $\mathcal{E}$  also lies in the case (9) of Theorem 1.1 by applying to  $\mathcal{E}$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if } p+q = 0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

We first claim that  $H^q(\mathcal{E}(-2)) = 0$  for all  $q$ . If  $n \geq 5$ , then this follows from (3.1) and (3.2). Suppose that  $n = 4$ . Then, by substituting  $c_2 = 5$  and  $c_3 = 7$  to (3.24), we have

$$0 \leq h^0(\mathcal{E}(-1)) = \frac{15 - c_4}{6}.$$

Hence  $c_4 \leq 15$  and  $c_4 \equiv 15 \pmod{6}$ . On the other hand, we have  $c_4 \leq 13$  by (3.25). Hence  $c_4 = 9$  or  $3$  by (3.7). This implies that  $H(\mathcal{E})^{r+3} \geq 4 > 0$ . Hence  $H^q(\mathcal{E}(-2)) = 0$  for all  $q$  by (3.1) and (3.3).

Note here that  $H^q(\mathcal{E}|_H(-1)) = 0$  if  $q > 0$  and that  $h^0(\mathcal{E}|_H(-1)) = 1$ . This implies that  $h^q(\mathcal{E}(-1)) = 0$  if  $q > 0$  and that  $h^0(\mathcal{E}(-1)) = 1$  by the claim above; consequently  $c_4 = 9$ . Note also that  $H^q(\mathcal{E}|_H(-k)) = 0$  for all  $q$  if  $2 \leq k \leq n - 2$ . This, together with the claim above, implies inductively that  $H^q(\mathcal{E}(-k)) = 0$  for all  $q$  if  $2 \leq k \leq n - 1$ . Finally note that  $H^q(\mathcal{E}|_H(1 - n)) = 0$  unless  $q = n - 2$  and that  $h^{n-2}(\mathcal{E}|_H(1 - n)) = 2$ . Since  $H^q(\mathcal{E}(1 - n)) = 0$  for all  $q$ , this implies that  $H^q(\mathcal{E}(-n)) = 0$  unless  $q = n - 1$  and that  $h^{n-1}(\mathcal{E}(-n)) = 2$ . Hence  $\text{Ext}^{n-1}(G, \mathcal{E}) \cong S_n^{\oplus 2}$  and  $\text{Ext}^q(G, \mathcal{E}) = 0$  unless  $q = n - 1$  or  $q = 0$ . Therefore it follows from (2.5) that  $E_2^{p,q} = 0$  unless  $(p, q) = (-n, n - 1)$  if  $q > 0$  and that  $E_2^{-n, n-1} \cong \mathcal{O}(-1)^{\oplus 2}$ .

The Bondal spectral sequence then implies that  $E_2^{p,0} = 0$  unless  $p = 0$  and that  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow E_2^{0,0} \rightarrow \mathcal{E} \rightarrow 0.$$

Hence  $E_2^{0,0}$  is a vector bundle. Since  $E_2^{-1,0} = 0$ , it follows from  $h^0(\mathcal{E}(-1)) = 1$  and (3.1) that  $E_2^{0,0}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus n+1} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E})$ . Now Lemma 4.3 shows that  $E_2^{0,0} \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus e_{0,0} - n - 1}$ . Hence  $e_{0,0} - n - 1 = r + 1$  and we obtain the case (9) of Theorem 1.1.

## 9. THE CASE WHERE $c_2(\mathcal{E}) = 6$

In this section, we give a proof of Theorem 1.1 in case  $c_2(\mathcal{E}) = 6$ ; throughout this section, we assume that  $c_2(\mathcal{E}) = 6$ .

**9.1. The case where  $n = 2$ .** Suppose that  $n = 2$ . Since  $c_2(\mathcal{E}) = 6$ , it follows from (3.10) and (3.12) that  $h^1(\mathcal{E}(-2)) = 3$  and  $h^0(\mathcal{E}) = 3 + r$ . Now we apply the Bondal spectral sequence (3.14).

If  $h^1(\mathcal{E}(-1)) \geq 1$ , then Lemma 5.1 (2) together with (3.15) and (3.16) shows that  $h^1(\mathcal{E}(-1)) = 1$ . Lemma 5.1 (1) then shows that  $E_2^{-2,1} \cong \mathcal{O}(-2)$  and that  $E_2^{-1,1} = 0$ . The exact sequence (3.16) then shows that  $\mathcal{E} \cong E_3^{0,0}$ , and thus  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \rightarrow E_2^{0,0} \rightarrow \mathcal{E} \rightarrow 0.$$

Hence  $E_2^{0,0}$  is a vector bundle. Since  $h^0(\mathcal{E}(-1)) = 1$  by (3.11), it follows from (3.17) that  $E_2^{0,0}$  also fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus 3} \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus 3+r} \rightarrow E_2^{0,0} \rightarrow 0.$$

Since  $E_2^{0,0}$  is a vector bundle, Lemma 4.3 then implies that  $E_2^{0,0} \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r}$ . Therefore  $\mathcal{E}$  is in the case (13) of Theorem 1.1.

If  $h^1(\mathcal{E}(-1)) = 0$ , then  $E_2^{-2,1} \cong \mathcal{O}(-1)^{\oplus 3}$  and  $E_2^{-1,1} = 0$  by (3.15). The exact sequence (3.16) then shows that  $\mathcal{E} \cong E_3^{0,0}$ , and thus  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus 3} \rightarrow E_2^{0,0} \rightarrow \mathcal{E} \rightarrow 0.$$

Moreover  $h^0(\mathcal{E}(-1)) = 0$  by (3.11). The exact sequence (3.17) then implies  $E_2^{0,0} \cong \mathcal{O}^{\oplus r+3}$ . Therefore  $\mathcal{E}$  is in the case (12) of Theorem 1.1.

**9.2. The case where  $n \geq 3$ .** Suppose first that  $n = 3$ . Since  $\mathcal{E}$  is nef, it follows from (3.23) that  $c_3 = c_3(\mathcal{E}) \geq 9$ . Hence  $\chi(\mathcal{E}(-1)) = (c_3/2) - 5 \geq 0$  by (3.20). This also implies that  $c_3 \geq 10$  and that  $c_3$  is even. Since  $c_3 > 9$ , we obtain  $H(\mathcal{E})^{r+2} > 0$ , and thus

$$(9.1) \quad H^q(\mathcal{E}(-1)) = 0 \text{ for all } q > 0$$

by (3.3). Hence  $\chi(\mathcal{E}(-1)) = h^0(\mathcal{E}(-1))$ . As we have seen in §9.1,  $h^0(\mathcal{E}|_{L^2}(-1)) \leq 1$  for any plane  $L^2$  in  $\mathbb{P}^3$ . It therefore follows from (3.1) that

$$(h^0(\mathcal{E}(-1)), h^0(\mathcal{E}|_{L^2}(-1)), c_3) = (0, 0, 10), (0, 1, 10), \text{ or } (1, 1, 12).$$

We claim here that if  $(h^0(\mathcal{E}(-1)), c_3) = (0, 10)$  then  $h^0(\mathcal{E}|_{L^2}(-1)) = 0$ . Suppose, to the contrary, that  $h^0(\mathcal{E}|_{L^2}(-1)) \neq 0$ . Then, as we have seen in §9.1,  $\mathcal{E}|_H$  lies in the case (13) of Theorem 1.1, and thus  $h^0(\mathcal{E}|_{L^2}(-1)) = 1$ ,  $h^1(\mathcal{E}|_{L^2}(-1)) = 1$ , and  $h^2(\mathcal{E}|_{L^2}(-1)) = 0$ . Hence  $h^1(\mathcal{E}(-2)) = 1$ ,  $h^2(\mathcal{E}(-2)) = 1$ , and  $h^3(\mathcal{E}(-2)) = 0$ . Therefore  $\text{Ext}^1(G, \mathcal{E}(1)) \cong S_3$ ,  $\text{Ext}^2(G, \mathcal{E}(1)) \cong S_3$ , and  $\text{Ext}^3(G, \mathcal{E}(1)) = 0$ . Moreover  $\text{Hom}(G, \mathcal{E}(1))$  has, as in (2.3), a projective resolution of the form

$$0 \rightarrow P_0^{\oplus 4e_{0,1}} \rightarrow P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}(1)) \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(1))$  and  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(1))$ . We apply to  $\mathcal{E}(1)$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q = 0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

We see that the  $E_2$  term  $E_2^{p,q}$  vanishes unless  $(p, q) = (-3, 2), (-3, 1), (-1, 0)$ , or  $(0, 0)$ , that  $E_2^{-3,2} \cong \mathcal{O}(-1)$ , and that  $E_2^{-3,1} \cong \mathcal{O}(-1)$ . The Bondal spectral sequence then implies that we have the following exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus 4e_{0,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0, \\ 0 \rightarrow \mathcal{O}(-1) \rightarrow E_2^{0,0} \rightarrow \mathcal{E}(1) \rightarrow 0. \end{aligned}$$

Since  $\text{Ext}^q(\mathcal{O}(-1), \mathcal{O}(-1)) = 0$  and  $\text{Ext}^q(\mathcal{O}(-1), \mathcal{O}^{\oplus e_{1,0}}) = 0$  for all  $q > 0$ , the sequences above induce the following exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus 4e_{0,1}} \oplus \mathcal{O}(-1) \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Lemma 4.4 then implies that  $4e_{0,1} \geq e_{0,0}$ . On the other hand, it follows from (3.21) and (3.22) that  $e_{0,1} = r + 3$  and  $e_{0,0} = 4r + 14$ , and thus  $4e_{0,1} - e_{0,0} = -2 < 0$ . This is a contradiction. Therefore the claim holds.

Suppose now that  $n \geq 3$ . Then we have two cases:  $c_3 = c_3(\mathcal{E}) = 10$  or  $12$  because  $c_3(\mathcal{E}) = c_3(\mathcal{E}|_{L^3}) = 10$  or  $12$  for a 3-dimensional linear subspace  $L^3 \subseteq \mathbb{P}^n$ .

9.2.1. *The case where  $c_3 = 10$ .* Suppose that  $c_3 = 10$ .

If  $n = 3$ , then we also have  $h^0(\mathcal{E}(-1)) = 0$  and  $h^0(\mathcal{E}|_{L^2}(-1)) = 0$ , as we have seen above. Hence  $H^q(\mathcal{E}(-1)) = 0$  for all  $q$  by (9.1) and  $\mathcal{E}|_{L^2}$  lies in the case (12) of Theorem 1.1, as we have seen in §9.1.

Suppose that  $n \geq 3$  and that the restriction  $\mathcal{E}|_H$  of  $\mathcal{E}$  to a hyperplane  $H$  in  $\mathbb{P}^n$  lies in the case (12) of Theorem 1.1. We shall show that  $\mathcal{E}$  also lies in the case (12) of Theorem 1.1 by applying to  $\mathcal{E}$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

We claim first that  $H^q(\mathcal{E}(-1)) = 0$  for all  $q$ ; the claim holds for  $n = 3$  as we have seen in the above. Suppose that  $n \geq 4$ . Since  $H^0(\mathcal{E}|_H(-1)) = 0$ , it follows from (3.1) that  $H^0(\mathcal{E}(-1)) = 0$ . Moreover  $H^q(\mathcal{E}(-1)) = 0$  if  $q > 0$  by (3.2) since  $n \geq 4$ . Hence the claim also holds for  $n \geq 4$ .

The claim above inductively (on  $k$ ) implies that  $H^q(\mathcal{E}(-k)) = 0$  for all  $q$  if  $1 \leq k \leq n-1$  since  $H^q(\mathcal{E}|_H(-k)) = 0$  for all  $q$  if  $1 \leq k \leq n-2$ . This implies that  $h^q(\mathcal{E}(-n)) = 0$  unless  $q = n-1$  and that  $h^{n-1}(\mathcal{E}(-n)) = 3$  since  $h^q(\mathcal{E}|_H(-n+1)) = 0$  unless  $q = n-2$  and  $h^{n-2}(\mathcal{E}|_H(-n+1)) = 3$ . Hence  $\text{Ext}^q(G, \mathcal{E}) = 0$  unless  $q = n-1$  or  $q = 0$ ,  $\text{Ext}^{n-1}(G, \mathcal{E}) \cong S_n^{\oplus 3}$ , and  $\text{Hom}(G, \mathcal{E}) \cong S_0^{\oplus e_{0,0}}$  where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E})$ . Therefore  $E_2^{p,q} = 0$  unless  $(p, q) = (-n, n-1)$  or  $(0, 0)$ ,  $E_2^{-n, n-1} \cong \mathcal{O}(-1)^{\oplus 3}$ , and  $E_2^{0,0} \cong \mathcal{O}^{\oplus e_{0,0}}$ . The Bondal spectral sequence then implies that  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus 3} \rightarrow \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E} \rightarrow 0.$$

Now we see that  $e_{0,0} = r + 3$  and that  $\mathcal{E}$  lies in the case (12) of Theorem 1.1.

9.2.2. *The case where  $c_3 = 12$ .* Suppose that  $c_3 = 12$ .

Suppose that  $n = 3$ . In this case, we also have  $h^0(\mathcal{E}(-1)) = 1$  and  $h^0(\mathcal{E}|_{L^2}(-1)) = 1$ ; in fact  $H^0(\mathcal{E}(-1)) \cong H^0(\mathcal{E}|_{L^2}(-1))$ . Thus  $h^1(\mathcal{E}(-2)) = 0$  by (9.1). Since  $h^0(\mathcal{E}|_{L^2}(-1)) = 1$ ,  $\mathcal{E}|_{L^2}$  lies in the case (13) of Theorem 1.1, as we have seen in §9.1. Hence we infer that  $h^1(\mathcal{E}|_{L^2}(-1)) = 1$  and that  $h^2(\mathcal{E}|_{L^2}(-1)) = 0$ . This, together with (9.1), implies that  $h^2(\mathcal{E}(-2)) = 1$  and  $h^3(\mathcal{E}(-2)) = 0$ . Summing up with (3.2) and (9.1), we conclude that if  $n = 3$  then

$$(9.2) \quad \text{Ext}^q(G, \mathcal{E}(1)) = 0 \text{ unless } q = 2 \text{ or } 0 \text{ and } \text{Ext}^2(G, \mathcal{E}(1)) \cong S_3.$$

Suppose that  $n \geq 3$  and that the restriction  $\mathcal{E}|_H$  of  $\mathcal{E}$  to a hyperplane  $H$  in  $\mathbb{P}^n$  lies in the case (13) of Theorem 1.1. Then we shall show that  $\mathcal{E}$  also lies in the case (13) of Theorem 1.1 by applying to  $\mathcal{E}(1)$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

We claim here that  $h^q(\mathcal{E}(-2)) = 0$  for all  $q$  if  $n \geq 4$ . If  $n \geq 5$ , this follows from (3.1) and (3.2). Suppose that  $n = 4$ . Then, by substituting  $c_2 = 6$  and  $c_3 = 12$  to (3.24), we have

$$0 \leq h^0(\mathcal{E}(-1)) = 5 - \frac{c_4}{6}.$$

Hence  $c_4 \leq 30$  and  $c_4 \equiv 0 \pmod{6}$ . On the other hand,  $c_4 \leq 27$  by (3.25). Thus  $c_4 \leq 24$ . Therefore  $H(\mathcal{E})^{r+3} \geq 3 > 0$ , so that  $h^q(\mathcal{E}(-2)) = 0$  for all  $q$  by (3.1) and (3.3).



The claim above then implies that if  $n \geq 4$  then  $h^q(\mathcal{E}(-1)) = 0$  for all  $q > 0$  and  $h^0(\mathcal{E}(-1)) = 1$ , since  $h^q(\mathcal{E}|_H(-1)) = 0$  for all  $q > 0$  and  $h^0(\mathcal{E}|_H(-1)) = 1$  if  $n \geq 4$ .

The claim above also inductively implies that  $H^q(\mathcal{E}(-k)) = 0$  for all  $q$  if  $2 \leq k \leq n-2$  and  $n \geq 4$ , since  $H^q(\mathcal{E}|_H(-k)) = 0$  for all  $q$  if  $2 \leq k \leq n-3$ .

Note that  $H^q(\mathcal{E}|_H(2-n)) = 0$  unless  $q = n-2$  and that  $h^{n-2}(\mathcal{E}|_H(2-n)) = 1$ . Since  $H^q(\mathcal{E}(2-n)) = 0$  for all  $q$  if  $n \geq 4$ , this implies that if  $n \geq 4$  then  $H^q(\mathcal{E}(1-n)) = 0$  unless  $q = n-1$  and  $h^{n-1}(\mathcal{E}(1-n)) = 1$ .

Summing these up with (9.2) and (3.2), we conclude that if  $n \geq 3$  then

$$\text{Ext}^q(G, \mathcal{E}(1)) = 0 \text{ unless } q = n-1 \text{ or } q = 0 \text{ and } \text{Ext}^{n-1}(G, \mathcal{E}(1)) \cong S_n.$$

Thus  $E_2^{p,q} = 0$  unless  $(p, q) = (-n, n-1)$  if  $q > 0$ , and  $E_2^{-n, n-1} \cong \mathcal{O}(-1)$  by (2.5). The Bondal spectral sequence then implies that  $E_2^{p,0} = 0$  for all  $p < 0$  and that  $\mathcal{E}(1)$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow E_2^{0,0} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since  $E_2^{p,0} = 0$  for all  $p < 0$ , it follows from (2.2) that  $E_2^{0,0}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus(n+1)^2} \rightarrow \mathcal{O}(1)^{\oplus n+1} \oplus \mathcal{O}^{\oplus e_{1,0}} \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(1))$ ,  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(1))$ , and  $e_{1,0} = (n+1)\{e_{0,1} + (n+2)/2\}$ . These two exact sequences induce the following exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}^{\oplus(n+1)^2} &\rightarrow \mathcal{O}(1)^{\oplus n+1} \oplus \mathcal{O}^{\oplus e_{1,0}} \oplus \mathcal{O}(-1) \\ &\rightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E}(1) \rightarrow 0. \end{aligned}$$

It follows from Lemma 4.4 that this sequence is reduced to the following exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus(n+1)^2} \rightarrow \mathcal{O}^{\oplus n+1} \oplus \mathcal{O}(-1)^{\oplus e_{1,0}-e_{0,0}} \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{E} \rightarrow 0.$$

By looking at  $\deg \mathcal{E}$ , we see that  $e_{1,0} - e_{0,0} = (n+1)^2$ . By looking at the rank of  $\mathcal{E}$ , we see that  $e_{0,1} = r + n + 1$ . Note that the composite  $\mathcal{O}(1) \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{E}$  is injective. Let  $\mathcal{G}$  be the cokernel of the injection  $\mathcal{O}(1) \rightarrow \mathcal{E}$ , and let  $\mathcal{H}$  be the kernel of the morphism  $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{E}$ . Then we have an exact sequence

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{G} \rightarrow 0.$$

Since  $\mathcal{H}$  is a subsheaf of  $\mathcal{O}^{\oplus e_{0,1}}$ , the kernel of the restriction to  $\mathcal{O}^{\oplus n+1}$  of the surjection  $\mathcal{O}^{\oplus n+1} \oplus \mathcal{O}(-1)^{\oplus e_{1,0}-e_{0,0}} \oplus \mathcal{O}(-2) \rightarrow \mathcal{H}$  is a direct sum of  $\mathcal{O}$ , and it must be contained in  $\mathcal{O}(-1)^{\oplus(n+1)^2}$ . Therefore it must be zero, i.e., the restriction to  $\mathcal{O}^{\oplus n+1}$  of the surjection  $\mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1)^{\oplus e_{1,0}-e_{0,0}} \oplus \mathcal{O}(-2) \rightarrow \mathcal{H}$  is injective. Therefore there exists a surjection  $\mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{O}^{\oplus n+1}$  such that the composite

$$\mathcal{O}^{\oplus n+1} \rightarrow \mathcal{O}^{\oplus n+1} \oplus \mathcal{O}(-1)^{\oplus e_{1,0}-e_{0,0}} \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{O}^{\oplus n+1}$$

is the identity. Hence the exact sequence above is reduced to an exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus(n+1)^2} \rightarrow \mathcal{O}(-1)^{\oplus(n+1)^2} \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r} \rightarrow \mathcal{E} \rightarrow 0.$$

This exact sequence is further reduced to an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r} \rightarrow \mathcal{E} \rightarrow 0.$$

This is the case (13) of Theorem 1.1.

10. THE CASE WHERE  $c_2(\mathcal{E}) = 7$ 

In this section, we give a proof of Theorem 1.1 in case  $c_2(\mathcal{E}) = 7$ ; throughout this section, we assume that  $c_2(\mathcal{E}) = 7$ .

**10.1. The case where  $n = 2$ .** Suppose that  $n = 2$ . Since  $c_2(\mathcal{E}) = 7$ , it follows from (3.10) that  $h^1(\mathcal{E}(-2)) = 4$ . It also follows from (3.11) that  $h^1(\mathcal{E}(-1)) = h^0(\mathcal{E}(-1)) + 1 \geq 1$ . Now we apply the Bondal spectral sequence (3.14); since  $h^1(\mathcal{E}(-2)) = 4$ , it follows from Lemma 5.1 (2) together with (3.15) and (3.16) that  $h^1(\mathcal{E}(-1)) = 1$ . Hence  $h^0(\mathcal{E}(-1)) = 0$ . Since  $h^1(\mathcal{E}(-2)) = 4$  and  $h^1(\mathcal{E}(-1)) = 1$ , Lemma 5.1 (1) then shows that  $E_2^{-2,1} \cong \mathcal{O}(-2) \oplus \mathcal{O}(-1)$  and that  $E_2^{-1,1} = 0$ . Hence  $\mathcal{E} \cong E_3^{0,0}$  by (3.16), and thus  $\mathcal{E}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \rightarrow E_2^{0,0} \rightarrow \mathcal{E} \rightarrow 0.$$

Note that  $h^0(\mathcal{E}) = r + 2$  by (3.12). Since  $h^0(\mathcal{E}(-1)) = 0$ , this implies that  $E_2^{0,0} \cong \mathcal{O}^{\oplus r+2}$  by (3.17). Therefore  $\mathcal{E}$  is in the case (14) of Theorem 1.1.

**10.2. The case where  $n \geq 3$ .** Suppose that  $n \geq 3$  and that the restriction  $\mathcal{E}|_H$  of  $\mathcal{E}$  to a hyperplane  $H$  in  $\mathbb{P}^n$  lies in the case (14) of Theorem 1.1. We shall show that  $\mathcal{E}$  also lies in the case (14) of Theorem 1.1 by applying to  $\mathcal{E}(1)$  the Bondal spectral sequence (2.1)

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p,q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

First, we claim that  $H^q(\mathcal{E}(-1)) = 0$  for all  $q$ . Since  $H^0(\mathcal{E}|_H(-1)) = 0$ , we first see that  $H^0(\mathcal{E}(-1)) = 0$  by (3.1). If  $n \geq 4$ , then  $H^q(\mathcal{E}(-1)) = 0$  for all  $q > 0$  by (3.2), and the claim holds. Suppose that  $n = 3$ . Note here that  $H^q(\mathcal{E}) = 0$  for all  $q > 0$  by (3.2) and that  $H^q(\mathcal{E}|_H) = 0$  for all  $q > 0$ . Hence  $H^q(\mathcal{E}(-1)) = 0$  unless  $q = 1$ . Thus

$$\chi(\mathcal{E}(-1)) = -h^1(\mathcal{E}(-1)) \leq 0.$$

On the other hand, Inequality (3.23) implies  $c_3 \geq 15$ . Hence

$$\chi(\mathcal{E}(-1)) = (c_3 - 15)/2 \geq 0$$

by (3.20). Therefore  $c_3 = 15$  and  $h^1(\mathcal{E}(-1)) = 0$ ; the claim also holds in case  $n = 3$ .

Secondly, we claim that  $H^q(\mathcal{E}(-k)) = 0$  for all  $q$  if  $1 \leq k \leq n-2$ ; this is nothing but the first claim if  $n = 3$ . Suppose that  $n \geq 4$  and that the claim holds for  $\mathcal{E}|_H$ : we have  $H^q(\mathcal{E}|_H(-k)) = 0$  for all  $q$  if  $1 \leq k \leq n-3$ . Then we see inductively on  $k$  that  $H^q(\mathcal{E}(-k)) = 0$  for all  $q$  if  $1 \leq k \leq n-2$ .

Thirdly, we claim that  $h^q(\mathcal{E}(1-n)) = 0$  unless  $q = n-1$  and that  $h^{n-1}(\mathcal{E}(1-n)) = 1$ . This claim is proved by induction on  $n$ : if  $n = 2$ , the claim holds; suppose that  $n \geq 3$  and that the claim holds for  $n-1$ ; now that  $h^q(\mathcal{E}(2-n)) = 0$  for all  $q$  by the second claim, the third claim follows from the induction hypothesis.

Now  $\text{Ext}^q(G, \mathcal{E}(1)) = 0$  unless  $q = n-1$  or  $q = 0$ ,  $\text{Ext}^{n-1}(G, \mathcal{E}(1)) \cong S_n$ , and the right  $A$ -module  $\text{Hom}(G, \mathcal{E}(1))$  has, as in (2.3), a projective resolution of the form

$$0 \rightarrow P_0^{\oplus(n+1)e_{0,1}} \rightarrow P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \rightarrow \text{Hom}(G, \mathcal{E}(1)) \rightarrow 0$$

where  $e_{0,0} = \text{hom}(\mathcal{O}, \mathcal{E}(1))$  and  $e_{0,1} = \text{hom}(\mathcal{O}(1), \mathcal{E}(1))$ . Therefore  $E_2^{p,q} = 0$  unless  $(p, q) = (-n, n-1)$ ,  $(-1, 0)$ , or  $(0, 0)$ , and  $E_2^{-n, n-1} \cong \mathcal{O}(-1)$  by (2.5). The Bondal spectral sequence then implies that  $E_2^{-1,0} = 0$  and that  $\mathcal{E}(1)$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow E_2^{0,0} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since  $E_2^{-1,0} = 0$ ,  $E_2^{0,0}$  fits in an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus(n+1)e_{0,1}} \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow E_2^{0,0} \rightarrow 0.$$

These two exact sequences then induce an exact sequence

$$0 \rightarrow \mathcal{O}^{\oplus(n+1)e_{0,1}} \oplus \mathcal{O}(-1) \rightarrow \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

It follows from Lemma 4.4 that this sequence is reduced to the following exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus(n+1)e_{0,1}-e_{0,0}} \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{E} \rightarrow 0.$$

By looking at  $\deg \mathcal{E}$ , we see that  $(n+1)e_{0,1} - e_{0,0} = 1$ . By looking at the rank of  $\mathcal{E}$ , we see that  $e_{0,1} = r+2$ ; thus this is the case (14) of Theorem 1.1.

## 11. A REMARK ON THE PROOF OF GLOBAL GENERATION IN THEOREM 1.1

**Proposition 11.1.** *Let  $\mathcal{E}$  be a nef vector bundle of rank  $r$  on a projective space  $\mathbb{P}^n$  with first Chern class  $c_1 = 3$  and second Chern class  $c_2 \leq 7$ . Then  $\mathcal{E}$  is globally generated unless  $(n, c_2, c_3) = (3, 5, 3)$ .*

*Proof.* Recall that  $\mathcal{E}$  is globally generated if  $h^1(\mathcal{E}(-1)) = 0$  and  $\mathcal{E}|_H$  is globally generated for any hyperplane  $H$  in  $\mathbb{P}^n$  by [PSW92, Lemma 3]. As we have seen in §§6.1, 7.1, 8.1, 9.1, and 10.1,  $\mathcal{E}$  is globally generated if  $n = 2$ . Suppose that  $n = 3$ . If  $h^1(\mathcal{E}(-1)) \neq 0$ , then it follows from (3.3) that  $H(\mathcal{E})$  is not big, and hence  $c_3 = 6c_2 - 27$  by (3.23). Since  $c_3$  is odd and non-negative by (3.7),  $c_2$  is odd by (3.20) and  $c_2 \geq 5$ ; thus  $(c_2, c_3) = (5, 3)$  or  $(7, 15)$ . However, as we have seen in §10.2,  $h^1(\mathcal{E}(-1)) = 0$  if  $c_2 = 7$ , so that the latter case is ruled out. Suppose that  $n \geq 4$ . Then  $h^1(\mathcal{E}(-1)) = 0$  by (3.2). Note that the restriction  $\mathcal{E}|_{L^3}$  to a 3-dimensional linear subspace  $L^3$  does not belong to the exceptional case, as we have seen in §8.3.1. Therefore  $\mathcal{E}$  is globally generated unless  $(n, c_2, c_3) = (3, 5, 3)$ .  $\square$

## 12. EXISTENCE OF NEF BUT NON-GLOBALLY GENERATED VECTOR BUNDLES

**Lemma 12.1.** *Let  $\mathcal{F}$  be a nef vector bundle on a smooth projective surface  $X$ . Let  $\mathcal{E}_0$  be a torsion-free quotient of  $\mathcal{F}$ , i.e., there exists a surjection  $\mathcal{F} \rightarrow \mathcal{E}_0$  with  $\mathcal{E}_0$  a torsion-free coherent sheaf. Let  $\mathcal{E}$  denote the double dual  $\mathcal{E}_0^{\vee\vee}$  of  $\mathcal{E}_0$ . Then  $\mathcal{E}$  is a nef vector bundle.*

*Proof.* Since  $\mathcal{E}$  is a reflexive sheaf on a smooth surface,  $\mathcal{E}$  is a vector bundle. To show that  $\mathcal{E}$  is nef, it is enough to show that, for any finite morphism  $C \rightarrow X$  from a smooth curve  $C$ , every quotient line bundle  $\mathcal{L}$  of  $\mathcal{E}|_C$  has non-negative degree. Note here that the natural injection  $\mathcal{E}_0 \rightarrow \mathcal{E}$  induces a generically injective morphism  $\mathcal{E}_0|_C \rightarrow \mathcal{E}|_C$ . Now let  $\mathcal{M}$  be the image of the composite of the morphism  $\mathcal{E}_0|_C \rightarrow \mathcal{E}|_C$  and the surjection  $\mathcal{E}|_C \rightarrow \mathcal{L}$ . Since the composite of  $\mathcal{F}|_C \rightarrow \mathcal{E}_0|_C$  and  $\mathcal{E}_0|_C \rightarrow \mathcal{M}$  is surjective and  $\mathcal{F}$  is nef, we see that  $\mathcal{M}$  has non-negative degree. Hence  $\mathcal{L}$  has non-negative degree since there is an injection  $\mathcal{M} \rightarrow \mathcal{L}$  of line bundles on the smooth curve  $C$ . Therefore  $\mathcal{E}$  is nef.  $\square$

As is indicated by the statement in Lemma 5.1 (2) (a), we can construct a nef but non-globally generated vector bundle on  $\mathbb{P}^2$  if  $c_1 = 3$  and  $c_2 = 8$ .

*Proof of Proposition 1.2.* Given an integer  $r \geq 2$  and a closed point  $w$  in  $\mathbb{P}^2$ , note first that there exists a section  $s$  in  $H^0(\mathcal{O}(3)^{\oplus r+1})$  such that the zero locus  $(s)_0$  of  $s$  is  $\{w\}$  as closed subschemes. Let  $\varphi : \mathcal{O}(-3) \rightarrow \mathcal{O}^{\oplus r+1}$  be the morphism determined by  $s$ , and  $\mathcal{E}_0$

the cokernel of  $\varphi$ . The dual  $\varphi^\vee : \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{O}(3)$  of  $\varphi$  has  $\mathcal{I}_w(3)$  as its image, where  $\mathcal{I}_w$  is the ideal sheaf of the point  $w$ , and we obtain an exact sequence

$$0 \rightarrow \mathcal{E}_0^\vee \rightarrow \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{I}_w(3) \rightarrow 0.$$

On the other hand, the ideal sheaf  $\mathcal{I}_w$  sits in an exact sequence

$$(12.1) \quad 0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow \mathcal{I}_w \rightarrow 0.$$

Therefore  $\mathrm{Tor}_i^{\mathcal{O}_w}(\mathcal{I}_w(3), k(w)) = 0$  for  $i > 1$ . Hence  $\mathcal{E}_0^\vee$  is a vector bundle. The exact sequence (12.1) also implies that  $\mathcal{E}xt^1(\mathcal{I}_w(3), \mathcal{O}) \cong k(w)$ . Let  $\mathcal{E}$  be the double dual  $\mathcal{E}_0^{\vee\vee}$  of  $\mathcal{E}_0$ . Since  $\mathcal{H}om(\mathcal{I}_w(3), \mathcal{O}) \cong \mathcal{O}(-3)$ , the vector bundle  $\mathcal{E}$  thus fits in the desired exact sequence

$$(12.2) \quad 0 \rightarrow \mathcal{O}(-3) \xrightarrow{\varphi} \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E} \rightarrow k(w) \rightarrow 0.$$

Suppose next that a vector bundle  $\mathcal{E}$  fits in the exact sequence (12.2). We split the sequence (12.2) into the following two exact sequences:

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E}_0 \rightarrow 0; \\ 0 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E} \rightarrow k(w) \rightarrow 0. \end{aligned}$$

We see that  $\mathcal{E}_0$  is a torsion-free sheaf of rank  $r$  with  $c_1(\mathcal{E}_0) = 3$ ,  $c_2(\mathcal{E}_0) = 9$ , and  $h^1(\mathcal{E}_0) = 1$ . We have  $\mathcal{E} \cong \mathcal{E}_0^{\vee\vee}$ , and thus  $\mathcal{E}$  is a nef vector bundle by Lemma 12.1. Moreover  $c_1(\mathcal{E}) = 3$  and  $c_2(\mathcal{E}) = 8$ . Since  $c_2(\mathcal{E}) = 8 < 9$ , we obtain  $h^1(\mathcal{E}) = 0$  by (3.8). Hence  $H^0(\mathcal{E}_0) \cong H^0(\mathcal{E})$ . Therefore  $\mathcal{E}$  is not globally generated.  $\square$

**Remark 12.2.** *If  $r = 2$ , the exact sequence in Proposition 1.2 already appears in [Lan98, 3.2.5]. Professor Adrian Langer kindly informed the author of this fact and that he ruled out this case by mistake.*

### 13. SOME EXAMPLES

Let  $X$  be a smooth projective variety, and  $\mathcal{E}$  a vector bundle on  $X$  of rank  $r$ . It is well known (see, e.g., [Băn91, Statement(folklore) 4.1]) that if  $\mathcal{E}$  is globally generated then general  $r - 1$  global sections of  $\mathcal{E}$  define an injection  $\mathcal{O}_X^{\oplus r-1} \rightarrow \mathcal{E}$  and this injection extends to an exact sequence

$$0 \rightarrow \mathcal{O}_X^{\oplus r-1} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \otimes \det \mathcal{E} \rightarrow 0$$

where  $\mathcal{I}_Z$  is the ideal sheaf of a locally complete intersection subscheme  $Z$  of, if not empty, codimension two in  $X$ . For nef vector bundles, however, analogous results do not hold in general, even if  $h^0(\mathcal{E}) \geq r - 1$ , as the following examples show.

**Example 13.1.** *Let  $\mathcal{E}_0$  be a nef vector bundle of rank  $r - 2$  on  $\mathbb{P}^2$  fitting in the following exact sequence*

$$0 \rightarrow \mathcal{O}(-4) \rightarrow \mathcal{O}^{\oplus r-1} \rightarrow \mathcal{E}_0 \rightarrow 0.$$

*Then  $r \geq 4$  and  $h^1(\mathcal{E}_0) = 3$ . Let  $\xi_1$  and  $\xi_2$  be linearly independent elements in  $H^1(\mathcal{E}_0)$ , and let*

$$0 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}^{\oplus 2} \rightarrow 0$$

*be the exact sequence whose extension class in  $\mathrm{Ext}^1(\mathcal{O}^{\oplus 2}, \mathcal{E}_0)$  is determined by  $\xi_1$  and  $\xi_2$ . Then the connecting homomorphism  $H^0(\mathcal{O}^{\oplus 2}) \rightarrow H^1(\mathcal{E}_0)$  is injective, and thus  $h^0(\mathcal{E}) = h^0(\mathcal{E}_0) = r - 1$ . Moreover  $\mathcal{E}$  is a nef vector bundle of rank  $r$  by [Laz04, Theorem 6.2.12 (ii)]. In this example, every morphism  $\mathcal{O}^{\oplus r-1} \rightarrow \mathcal{E}$  is not injective.*

**Example 13.2.** Let  $\mathcal{E}_0$  be a nef vector bundle of rank  $r - 1$  on  $\mathbb{P}^2$  fitting in the following exact sequence

$$0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O}^{\oplus r} \rightarrow \mathcal{E}_0 \rightarrow 0.$$

Then  $r \geq 3$  and  $h^1(\mathcal{E}_0) = 1$ . Let

$$0 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E} \rightarrow \mathcal{O} \rightarrow 0$$

be a non-split exact sequence; the connecting homomorphism  $H^0(\mathcal{O}) \rightarrow H^1(\mathcal{E}_0)$  is an isomorphism. Then  $h^0(\mathcal{E}) = h^0(\mathcal{E}_0) = r$ , and it follows from [Laz04, Theorem 6.2.12 (ii)] that  $\mathcal{E}$  is a nef vector bundle of rank  $r$  with  $c_1 = 3$  and  $c_2 = 9$ . In this example, a general morphism  $\mathcal{O}^{\oplus r-1} \rightarrow \mathcal{E}$  is injective, but its cokernel  $\mathcal{C}$  fits in a non-split exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{C} \rightarrow \mathcal{O} \rightarrow 0$$

where  $\mathcal{O}_C$  is the structure sheaf of some curve  $C$  of degree 3 in  $\mathbb{P}^2$ ; since  $\mathcal{C}$  has a non-zero torsion subsheaf  $\mathcal{O}_C$ ,  $\mathcal{C}$  is not isomorphic to a torsion-free coherent sheaf  $\mathcal{I}_Z \otimes \det \mathcal{E}$  for any closed subscheme  $Z$  of  $\mathbb{P}^2$ .

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